Maxwell's eq$^n$ are linear

$\frac{d}{dt}(U_{mech}) = \vec{J}\cdot\vec{E}$

$\vec{D} = \epsilon \vec{E}$

$\vec{B} = \frac{1}{\mu}\vec{H}$

1 $\nabla\cdot\vec{D} = 4\pi\rho$

2 $\nabla\cdot\vec{B} = 0$

3 $\nabla\wedge\vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}(\vec{B})$

4 $\nabla\wedge\vec{H} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial}{\partial t}(\vec{D})$

density

$\partial_t \rho + \nabla\cdot F = 0$ "stuff" is conserved

no trivial sol$^{ns}$s

in vaccuum: $\epsilon = \mu = 1$

divergence of curl = 0

$\nabla$ of 4 $\quad \frac{4\pi}{c}\nabla\vec{J} + \frac{1}{c}\partial_t(\underbrace{\nabla\vec{D}}_{\to 1})$

$= \frac{4\pi}{c}\left(\nabla\vec{J} + \partial_t(\rho)\right) = 0$

$\longrightarrow$ charge is conserved

$\vec{J}\cdot\vec{E}$ = rate of work = $\frac{d}{dt}(U_{mech})$

$\underbrace{\vec{J}}_{4}\cdot\vec{E} = \frac{1}{4\pi}\left(\overbrace{c(\nabla H)}^{\text{curl of } H}\cdot\vec{E} - \vec{E}\cdot\partial_t(\vec{D})\right)$

$\vec{E}\cdot\nabla\wedge\vec{H} = \underbrace{\vec{H}\,\nabla\wedge\vec{E}}_{\to 3} - \nabla(\vec{E}\times\vec{H})$ some identity

$\vec{J}\cdot\vec{E} + \frac{1}{8\pi}\partial_t\left(\epsilon\vec{E}^2 + \frac{1}{\mu}\vec{B}^2\right) = -\nabla\frac{c}{4\pi}(\vec{E}\wedge\vec{H})$

divergence of flux

$U_{EM} = \frac{1}{8\pi}\underbrace{\left[\epsilon\vec{E}^2 + \frac{1}{\mu}\vec{B}^2\right]}_{\text{EM energy density}}$

$\vec{S} = \frac{c}{4\pi}(\vec{E}\wedge\vec{H})$ Poyntin Vector

$\longrightarrow \frac{\partial}{\partial t}(U_{Mech} + U_{EM}) + \nabla\cdot\vec{S} = 0$

$\frac{d}{dt}\int_V (U_{mech} + U_{EM})\,dV = -\int_\Sigma \vec{S}\cdot\vec{n}\,dA$

grows @ $r^{-4}$

contribution to energy small @ large

So why can we see stars? lol they're not electrostatic

Divergence of curl: $\nabla(E \times H)$

$\vec{E} \equiv E_i$

$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{else} \end{cases}$

$\nabla \cdot \vec{E} = \partial_i E_i$

$\epsilon_{ijk} = \begin{cases} 1 & 123 \text{ or even perm} \\ -1 & 132 \text{ or odd perm} \\ 0 & \text{else} \end{cases}$

$[\vec{A} \times \vec{B}]_i = \epsilon_{ijk} A_j B_k$

$$(\nabla [E \times H])_i = \partial_i [\epsilon_{ijk} E_j H_k]$$

$$= \epsilon_{ijk} [E_j \partial_i(H_k) + H_k \partial_i(E_j)]$$

$$= -\epsilon_{jik} E_j \partial_i(H_k) + \epsilon_{kij} H_k \partial_i(E_j)$$

$$= -E_j \epsilon_{jik} \partial_i(H_k) + H_k \epsilon_{kij} \partial_i(E_j)$$

$$= -\vec{E}(\nabla \wedge \vec{H} + \vec{H} \nabla \wedge \vec{E})$$

$\epsilon_{ijk} \epsilon_{lmn} = $ ?

$\nabla \wedge \nabla \wedge = \nabla(\nabla \quad ) - \nabla^2$

Vaccuum Solution

$\nabla \cdot \vec{E} = \nabla \cdot \vec{B} = 0$ 5

$\nabla \wedge \vec{E} = -\frac{1}{c} \partial_t(\vec{B})$ 6

$\nabla \wedge \vec{B} = \frac{1}{c} \partial_t(\vec{E})$ 7

6 $\rightarrow$ $\nabla \wedge (\nabla \wedge \vec{E}) = \nabla(\nabla \vec{E}) - \nabla \vec{E}^2 = -\frac{1}{c} \partial_t \nabla \wedge B$ 7

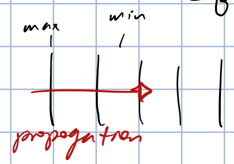
$$= -\frac{1}{c^2} \partial_t^2 (\vec{E})$$

$\nabla^2 \vec{E} - \frac{1}{c^2} \partial_t^2(\vec{E}) = 0$

$\nabla^2 \vec{B} - \frac{1}{c^2} \partial_t^2(\vec{B}) = 0$

# Simplest nontrivial sol$^n$ of wave eq$^n$

Plane waves




$$\vec{E} = \vec{a}_1 E_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$|\vec{a}_1| = |\vec{a}_2| = 1$$

$$\vec{B} = \vec{a}_2 B_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$\vec{k} \to$ wave number

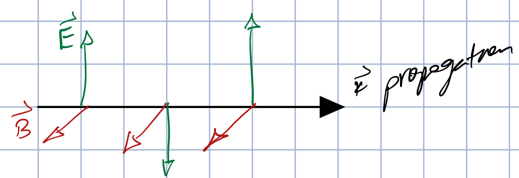
$\omega \to$ frequency

$E_0, B_0 \in \mathbb{C}$

$$\nabla \cdot \vec{E} = 0 = \cancel{i}\,\vec{k}\cdot\vec{a}_1 E_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

either $E_0$ & $B_0 = 0$ or $\vec{k}\cdot\vec{a}_1 E_0 = 0$ & $\vec{k}\cdot\vec{a}_2 B_0 = 0$

$\vec{k}$ is $\perp$ to $\vec{a}_1$ & $\vec{a}_2$




EM waves are transverse

$\vec{E}$ is perp to $\vec{a}_1$ & $\vec{k}$

$$\cdot\; i\vec{k} \times \vec{a}_1 E_0 = i\frac{\omega}{c}\vec{a}_2 B_0 \qquad \text{from } 6$$

"jolly good"

$$i\vec{k} \times \vec{a}_2 B_0 = -i\frac{\omega}{c}\vec{a}_1 E_0 \qquad \text{from } 7$$

$\vec{B}$ is perp to $\vec{a}_2$ & $k$

conclusion: $\vec{a}_1, \vec{a}_2, \vec{k}$ are orthogonal

$\cdot\vec{a}_1$

$$E_0 = \frac{\omega}{kc} B_0$$

$\cdot\vec{a}_2$

$$B_0 = \frac{\omega}{kc} E_0$$

$$\xrightarrow{\omega = ck} E_0 = B_0$$

phase velocity of wave is $c$

$$V_{phase} = \frac{\omega}{k} = c$$

$$V_{group} = \frac{\partial\omega}{\partial k} = c$$

EM waves are nondispersive (all frequencies travel @ same speed)

Construct the time average

$$\langle S \rangle = \text{avg flux} = \frac{c}{4\pi} \langle \vec{E} \wedge \vec{B} \rangle = \frac{c}{8\pi} Re(E_0 B_0^*) = \frac{c}{8\pi} Re(E_0^* B_0)$$

energy density/velocity

$$= \frac{c}{8\pi} |E_0|^2 = \frac{c}{8\pi} |B_0|^2$$

$$\langle U \rangle = \frac{1}{8\pi} \langle E^2 + B^2 \rangle = \frac{1}{8\pi} |E_0|^2 = \frac{1}{8\pi} |B_0|^2$$

energy density

$$\frac{\langle S \rangle}{\langle U \rangle} = c$$

No mechanical exp. can say if you're in motion

$\longrightarrow$ ha! try to measure light irrespective of medium

they thought light's propogation material is ether