Shor Code

$$a|0\rangle + b|1\rangle \longrightarrow \frac{a}{2\sqrt{2}}(|000\rangle + |111\rangle)^{\otimes 3} + \frac{b}{2\sqrt{2}}(|000\rangle - |111\rangle)^{\otimes 3}$$

phase flip in 1^{st} block

$$|\Psi_E\rangle = \frac{a}{2\sqrt{2}}(|000\rangle - |111\rangle)\otimes(|000\rangle + |111\rangle)^{\otimes 2} - \frac{b}{2\sqrt{2}}(|000\rangle + |111\rangle)\otimes(|000\rangle - |111\rangle)^{\otimes 2}$$

measure $X_1 \dots X_6$ $\quad -1 \longrightarrow$ phase flip in 1^{st} of 3 blocks

$X_4 \dots X_9$ $\quad +1$

$(X_1 X_2 X_3)(X_4 X_5 X_6)$

phase flip in 1^{st} block, not specific bit

Correct w/ $Z_1 Z_2 Z_3$

$|000\rangle \rightarrow |000\rangle$

$|111\rangle \rightarrow -|111\rangle$ in 1^{st} block

set of operators need to measure to detect & correct error

$Z_1Z_2 \quad Z_2Z_3 \quad Z_3Z_4 \quad Z_4Z_5 \quad Z_5Z_6 \quad Z_{67} \quad Z_{78} \quad Z_{89}$ } all commute

$X_1, \dots, X_6$, $X_4 \dots X_9$

CSS codes - def^n & 1 example - connect to classical error correction

classical codes $\mathcal{C}_1$ & $\mathcal{C}_2$ - binary linear codes

$\mathcal{C}_1$ $[n, k_1]$ code $\quad k_1$ dimensional subspace of $(\mathbb{F}^2)^{\otimes n}$

$\mathcal{C}_2$ $[n, k_2]$ code $\quad k_2$ dimensional subspace of $(\mathbb{F}^2)^{\otimes n}$

s.t. $\mathcal{C}_1 \subset \mathcal{C}_2$ & $\mathcal{C}_1$ & $\mathcal{C}_2^{\perp}$ can both correct t errors

then the CSS code $CSS(\mathcal{C}_1, \mathcal{C}_2)$ is vector of subspace (of $(\mathbb{F}^2)^{\otimes n}$) spanned by states

$$|x + \mathcal{C}_2\rangle = \frac{1}{\sqrt{|\mathcal{C}_2|}} \sum_{y \in \mathcal{C}_2} |x+y\rangle$$

$$x \in \mathcal{C}_1$$

can show this can detect & correct t quantum errors example (Steane code)

$\mathcal{C}_1 = [7,4]$
$\mathcal{C}_2 = \mathcal{C}_1^\perp = [7,3]$, $t=1$

$CSS(\mathcal{C}_1, \mathcal{C}_2)$ is a $[7,1]$ Quantum Error Correction Code

Hamming code
1 word all 0's (0 00 0000) } even parity
1 word all 1's (1 1 1 1 1 1 1)
7 words w/3 1's (0000 111), (...), ... } odd parity
7 words w/4 1's (000 1111), (...), ...

$$|\bar{0}\rangle = \frac{1}{\sqrt{8}} \sum_{\text{even}} |w_1, w_2, \dots w_7\rangle$$

$w_1, w_2, \dots \in [7,4]$

$$|\bar{1}\rangle = \frac{1}{\sqrt{8}} \sum_{\text{odd}} |w_1, w_2, \dots, w_7\rangle$$

Next: ① why these codes work? ② Group theory cast ③ Toric code

① $a|0\rangle + b|1\rangle \xrightarrow{?} a|00\rangle + b|11\rangle$

bit flip 1st bit $|\psi_E\rangle = a|10\rangle + b|01\rangle$
2nd bit $|\psi_E\rangle = a|01\rangle + b|10\rangle$ not orthogonal

with 3 qubits, all 1 qubit in $|\psi_E\rangle$ were all orthogonal

Projection Operators onto orthogonal subspaces commute

$\longrightarrow$ measuring these projection operators allow to detect & correct errors

② For one qubit, we create / correct errors w/ X Y Z

taking products of these in all possible ways gives Pauli group

$G_1 = \{\pm \mathbb{1}, \pm i\mathbb{1}, \pm X, \pm i X, \pm Y, \pm i Y, \pm Z, \pm i Z\}$

On $(\mathbb{C}^2)^{\otimes n}$ we have group G_n of all products X_i, Y_j, Z_k $i,j,k = 1,..n$

Suppose S is a subgroup of Pauli group G_n

$$V_S = \{ |\psi\rangle \in (\mathbb{C}^2)^{\otimes n} \mid s|\psi\rangle = |\psi\rangle \text{ for all } s \in S\}$$

V_S – space stabilized by S

S – stabilizer of V_S

For G^n, can show that for V_S to be nontrivial,

$-\mathbb{1} \notin S$

S should be Abelian (all S elements commute)

3 qubit flipcode has $S = \{\mathbb{1}, Z_1Z_2, Z_2Z_3, Z_1Z_3\}$

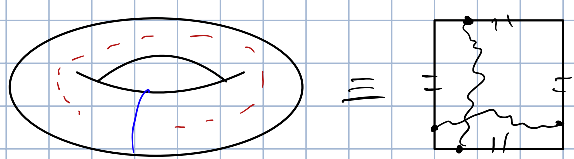
Steane Code is stabilizer code, S generated by 6 elements

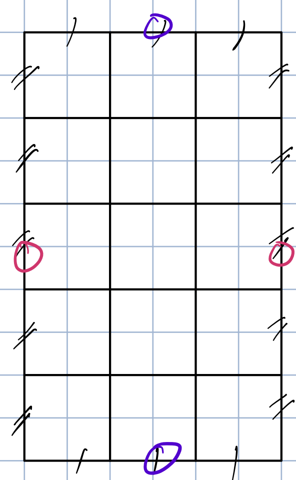
$Z|0\rangle = |0\rangle$ $\quad Z_{12}|000\rangle = |000\rangle$

$Z|1\rangle = -|1\rangle$ $\quad Z_{12}|111\rangle = (-1)(-1)|111\rangle = |111\rangle$

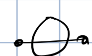
③ Toric code

2 Taurus





associate each edge → spin ½ particle or 2 state system



2 state point