# Quantum Error Correction

P. Shor 1995
Steane 1996

We will encode $|\psi\rangle = a|0\rangle + b|1\rangle \in \mathbb{C}^2$ into a state in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$

$$|\bar{\Psi}\rangle = a\,|0\rangle \otimes |0\rangle \otimes |0\rangle + b\,|1\rangle \otimes |1\rangle \otimes |1\rangle$$
$$= a\,|000\rangle + b\,|111\rangle$$

tripling components, not cloning

NOT $(a|0\rangle + b|1\rangle) \otimes (a|0\rangle + b|1\rangle) \otimes (a|0\rangle + b|1\rangle)$

Focus on correcting a one bit flip error which takes $|0\rangle \to |1\rangle$ or $|1\rangle \to |0\rangle$ in one of the $\mathbb{C}^2$ components

2 step process

1. Detect a possible error through measurement that does not change $|\bar{\Psi}\rangle$
2. From the data of this measurement we will correct error

suppose bit flip error in $2^{nd}$ qubit

$$|\bar{\Psi}\rangle \to |\psi_E\rangle = a|010\rangle + b|101\rangle$$

Let's measure first qubit

projection operator  identity  identity

$$P = |1\rangle_1\langle 1|_1 \otimes \mathbb{1}_2 \otimes \mathbb{1}_3$$

$$\langle\psi_E|P|\psi_E\rangle = \langle\psi_E|\,(|1\rangle_1\langle 1|_1) \otimes \mathbb{1}_2 \otimes \mathbb{1}_3\,|\psi_E\rangle = |b|^2$$

after measurement, state is now in $b|101\rangle \neq |\bar{\Psi}\rangle$

define 4 Projection operators

$$P_0 = |000\rangle\langle 000| + |111\rangle\langle 111|$$
$$P_1 = |100\rangle\langle 100| + |011\rangle\langle 011|$$
$$P_2 = |010\rangle\langle 010| + |101\rangle\langle 101|$$
$$P_3 = |001\rangle\langle 001| + |110\rangle\langle 110|$$

can check

$$P_i^2 = P_i \quad \& \quad [P_i, P_j] = 0$$

$|\Psi_E\rangle = |\Psi\rangle$ $\quad |\Psi_E\rangle = a|100\rangle + b|011\rangle$

eigenvalues

| | No error | 1st Qubit | 2nd Qubit | 3rd Qubit |
|---|---|---|---|---|
| $P_0$ | 1 | 0 | 0 | 0 |
| $P_1$ | 0 | 1 | 0 | 0 |
| $P_2$ | 0 | 0 | 1 | 0 |
| $P_3$ | 0 | 0 | 0 | 1 |

$|\Psi\rangle$ & $|\Psi_E\rangle$ for a 1 qubit error are eigenstates of $P_0, P_1, P_2, P_3$

if there is a bit flip in ith qubit apply

$\sigma_x$ in ith pos.

$\mathbb{X}_i \qquad \mathbb{X}_2 = \mathbb{1} \otimes \sigma_x \otimes \mathbb{1}$

if $|\Psi_E\rangle = a|010\rangle + b|101\rangle$ (2nd qubit flip)

$$\mathbb{X}_2 |\Psi_E\rangle = (\mathbb{1} \otimes \sigma_x \otimes \mathbb{1}) |\Psi_E\rangle = |\Psi\rangle$$

What happens w/2 qubits?

$|\Psi\rangle = a|00\rangle + b|11\rangle$

$P_0 = |00\rangle\langle 00| + |11\rangle\langle 11|$

$P_1 = |10\rangle\langle 10| + |01\rangle\langle 01|$

flip 1st $\qquad$ flip 2nd

suppose we have $|\Psi_{E,1}\rangle = a|10\rangle + b|01\rangle$ & $|\Psi_{E,2}\rangle = a|01\rangle + b|10\rangle$

$P_0|\Psi_{E,1}\rangle = 0 = P_0|\Psi_{E,2}\rangle$

$P_1|\Psi_{E,1}\rangle = |\Psi_{E,1}\rangle$ $\qquad 1 + i\theta \mathbb{X}_2 - \frac{\theta^2}{2}\mathbb{X}_2^2 + \dots$

$P_1|\Psi_{E,2}\rangle = |\Psi_{E,2}\rangle$

$X_2^2 = (1 \otimes \sigma_x \otimes 1)^2 = 1$

What about continuous bit flip errors

$$|\psi_E\rangle = e^{i\theta X_i} |\Psi\rangle$$

$$e^{i\theta X_2} |\Psi\rangle$$

$$= (\cos\theta \, 1 + i\sin\theta \, X_2) |\Psi\rangle$$

$$= \cos\theta |\Psi\rangle + i\sin\theta \, (a|010\rangle + b|101\rangle)$$

$$\langle\psi_E|P_1|\psi_E\rangle = \langle\psi_E|P_3|\psi_E\rangle = 0$$

$$\langle\psi_E|P_0|\psi_E\rangle = \cos^2\theta \, (|a|^2 + |b|^2) = \cos^2\theta$$

$$\langle\psi_E|P_2|\psi_E\rangle = \sin^2\theta \, (|a|^2 + |b|^2) = \sin^2\theta$$