back at it

final is take home ~2 hours in 4 hour windows
working on practice problems for final
updated TeX Notes (time perturbation, Toric)
slight emphasis on 2^{nd} half

SHO

Harmonic Perturbations

oscillates w/ fixed frequency

$$\lambda H^{(1)}(t) = V^{+} e^{i\omega t} + V e^{-i\omega t}$$

assume V has small parameter that plays role of λ

initial state $|i\rangle$ s.t. $H^{(0)}|i\rangle = E_i|i\rangle$

final state $|f\rangle$ s.t. $H^{(0)}|f\rangle = E_f|f\rangle$

assume $|i\rangle \neq |f\rangle$ to look @ transitions $E_i \neq E_f$

$$P_{i\to f} = \frac{1}{\hbar^2}\left|\int_0^t \langle f| V^{+}e^{i\omega t'} + Ve^{-i\omega t'}|i\rangle e^{i\omega_{fi}t'}\,dt'\right|^2 \qquad \omega_{fi} = \frac{E_f - E_i}{\hbar}$$

$|c_n^{(1)}|^2 \quad n=f$

① $\int dt'$ ② matrix elements ③ final states containing free particle (photoelectric effect $2p \to 1s + \gamma$)

① $\int dt'$

$$\alpha_{\pm} = \omega_{fi} \pm \omega$$

$$\int_0^t e^{i(\omega_{fi}\pm\omega)t'}\,dt' = \left.\frac{e^{i(\omega_{fi}\pm\omega)t'}}{i(\omega_{fi}\pm\omega)}\right|_{t=0}^{t} = \frac{e^{i(\omega_{fi}\pm\omega)t} - e^{0}}{i(\omega_{fi}\pm\omega)} = \frac{e^{i\alpha_{\pm}t}-1}{i\alpha_{\pm}}$$

factor out trigs

$$= \left(\frac{e^{i\alpha_{\pm}t/2}}{\alpha_{\pm}/2}\right)\cdot\left(\frac{e^{i\alpha_{\pm}t/2} - e^{-i\alpha_{\pm}t/2}}{2i}\right) = e^{i\alpha_{\pm}t/2}\,\frac{\sin\left(\frac{\alpha_{\pm}t}{2}\right)}{\alpha_{\pm}/2}$$

$$P_{i\to f} = \frac{1}{\hbar^2}\left|\;\cdots\;\right|^2$$

$$= \frac{1}{\hbar^2}\left|\langle f|V^{+}|i\rangle e^{i\alpha_{+}t/2}\underbrace{\frac{\sin(\alpha_{+}t/2)}{\alpha_{+}/2}}_{*} + \langle f|V|i\rangle e^{i\alpha_{-}t/2}\underbrace{\frac{\sin(\alpha_{-}t/2)}{\alpha_{-}/2}}_{*}\right|^2$$

For $t \to$ small

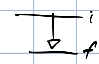
$$\sin(\alpha_{\pm}t/2) \approx \alpha_{\pm}t/2 + \ldots \qquad e^{i\alpha_{\pm}t/2} \to 1$$

$$P_{i\to f} = \frac{1}{\hbar^2}\left|\langle f|V^{+}|i\rangle\cdot t + \langle f|V|i\rangle t\right|^2 = t^2\left(\frac{1}{\hbar^2}\left|\langle f|V^{+}|i\rangle + \langle f|V|i\rangle\right|^2\right)$$

For large t & $\alpha_\pm$ small

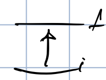
α_+ is small $\qquad \alpha_+ = \omega_{fi} + \omega = \frac{E_f - E_i}{\hbar} + \omega$

$\alpha_+ \to 0 \implies E_f - E_i + \hbar\omega \to 0 \implies E_f = E_i - \hbar\omega$



lose $\hbar\omega$

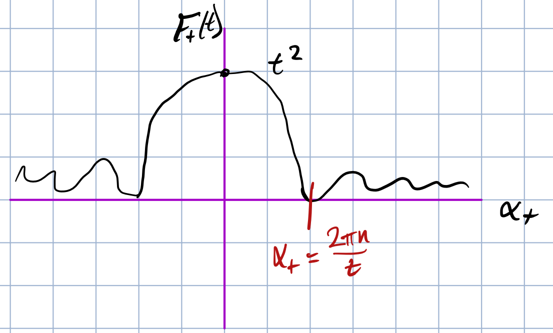
$\alpha_- \to 0 \implies E_f = E_i + \hbar\omega$



add $\hbar\omega$

when α_+ is small $\quad$ ★ dominates $\to$ $P_{i\to f} = \frac{1}{\hbar^2} |\langle f|V^+|i\rangle|^2 \cdot F_+(t)$

$$F_+(t) = \frac{4\sin^2(\alpha_+ t/2)}{\alpha_+^2}$$



$\alpha_+ \to 0, \quad F_+(t) \approx \frac{4(\alpha_+ t/2)^2}{\alpha_+^2} = t^2$

width $\sim \frac{2\pi n}{t}$

height $\sim t^2$

as $t \to \infty$

$\hookrightarrow \lim_{t\to\infty}\left(\frac{4}{\alpha_+^2}\cdot \sin^2\left(\frac{\alpha_+ t}{2}\right)\right) = 2\pi t\,\delta(\alpha_+) = 2\pi t\hbar\,\delta(E_f - E_i + \hbar\omega)$

$P_{i\to f} = \frac{1}{\hbar^2}|\langle f|V^+|i\rangle|^2 \cdot 2\pi t\,\delta(E_f - E_i + \hbar\omega) \qquad$ for small α_+

① delta f^n (it's ok, will integrate over energy range)

② linear in time

usually talk about rates of P_i

$$\lim_{t\to\infty}\left(\frac{P_{i\to f}}{t}\right) = \frac{1}{\hbar^2}|\langle f|V^+|i\rangle|^2\, 2\pi\hbar\,\delta(E_f - E_i + \hbar\omega)$$

"Fermi's" Golden Rule

$$\lim_{t\to\infty}\left(\frac{P_{i\to f}}{t}\right) = \frac{1}{\hbar^2}|\langle f|V^+|i\rangle|^2\, 2\pi\hbar\,\delta(E_f - E_i + \hbar\omega) \qquad \alpha_+ \to 0$$

$$\lim_{t\to\infty}\left(\frac{P_{i\to f}}{t}\right) = \frac{1}{\hbar^2}|\langle f|V|i\rangle|^2\, 2\pi\hbar\,\delta(E_f - E_i - \hbar\omega) \qquad \alpha_- \to 0$$

Final state & Counting States

put free particle in final state inside of a box $V=L^3$ w/ boundary conditions

free particle like electron $\psi(\vec{x})$

we know plane wave sol^n's: $\psi(\vec{x}) = e^{i\vec{p}\cdot\vec{x}}$ for any $\vec{p}$ that is a momentum eigenstate

$$\int_{box} \psi^*(\vec{x})\,\psi(\vec{x})\, d^3\vec{x} = \int_0^L dx \int_0^L dy \int_0^L dz\; |\psi(\vec{x})|^2 = 1$$

periodic boundary conditions: $\psi(x+L, y, z) = \psi(x, y+L, z) = \psi(x, y, z+L)$

$$\psi(\vec{x}) = \frac{e^{i\vec{p}\cdot\vec{x}/\hbar}}{\sqrt{V}} \qquad \vec{p} = \frac{2\pi\,\vec{n}}{L} \qquad \vec{n} = (n_x, n_y, n_z), \quad n_i \in \mathbb{Z}$$

we want to count states b/c we will sum over them

each choice $n = (n_x, n_y, n_z)$ gives state

summing free final states: $\sum_{n_x n_y n_z} P_{i\to f}$ with $f \leftarrow n_x, n_y, n_z$

as $L \to \infty$, $\sum_{n_x n_y n_z} \to \int d^3n = L^3 \int \frac{d^3p}{(2\pi\hbar)^3} = V \int \frac{d^3p}{(2\pi\hbar)^3}$

$$\Gamma_{i\to f} = \lim_{t\to\infty}\left(\frac{P_{i\to f}}{t}\right) = \frac{2\pi}{\hbar}\int \frac{d^3p}{(2\pi\hbar)^3}\cdot |M_{fi}|^2\, \delta(E_f(p) - E_i + \hbar\omega)$$

$$= \frac{2\pi}{\hbar}\, |M_{fi}|^2\, \underbrace{\rho(E)}_{\text{density of states}}$$

density of states,
also Fermi's golden rule

$M_{fi} = \langle f|V|i\rangle$ emmiting energy $\hbar\omega$

$M_{fi} = \langle f|V^{+}|i\rangle$ absorbing energy $\hbar\omega$