Fun w/ magnetic monopoles (just fun)

① Is QM consistent w/ long-range magnetic monopole fields?

$$\vec{B} = \frac{g}{r^2}\,\vec{e}_r \qquad \vec{\nabla}\cdot\vec{B} \neq 0$$

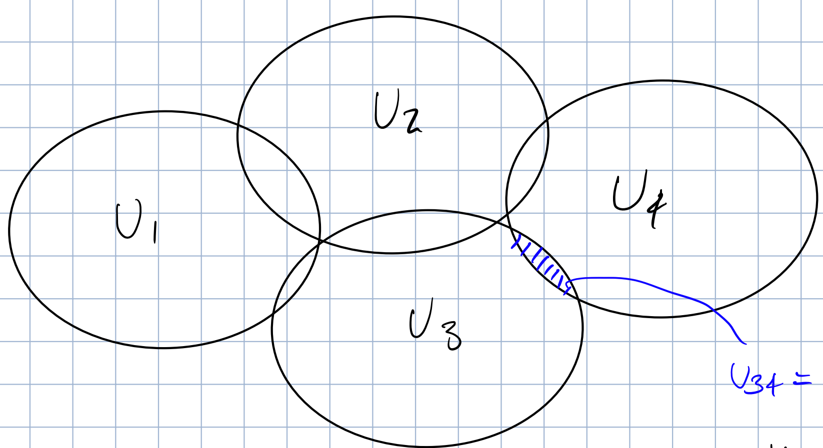
② Do actual theorems predict existence of magnetic monopoles?

↳ need string theory, not today
only talk about Q1

Standard Model? No
Grand Unified Theory? yes
Kaluza-Klein Theory? yes
String Theory? yes

Fiber bundle

a flexible framework for gauge transformations & $\vec{B}$ fields ($\vec{E}=0$, $\Phi=0$) can be included



intersection

$U_{34} = U_3 \cap U_4$

in each U_i, will have vector potential $\vec{A}_i$, a wave f^n $|\psi_i\rangle$, & $\vec{B}_i = \vec{\nabla}\times\vec{A}_i$

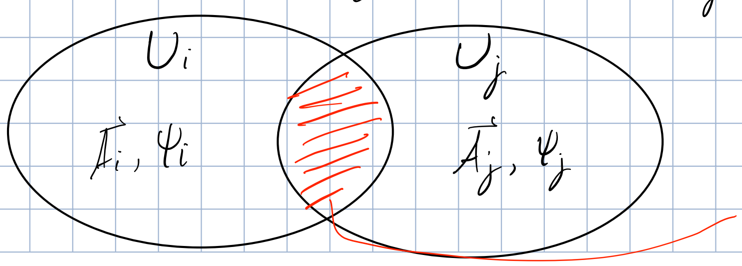
we want a well defined $\vec{B}$ everywhere

if $U_{ij} = U_i \cap U_j$. then U_{ij} better have $\vec{B}_i = \vec{\nabla}\times\vec{A}_i = \vec{B}_j = \vec{\nabla}\times\vec{A}_j$

$\Longrightarrow$ $\vec{A}_i$ & $\vec{A}_j$ are related by gauge transformation $\rightarrow$ $\vec{A}_i = \vec{A}_j + \vec{\nabla}\Lambda_{ij}$

in QM, gauge transformations also work on wave f^n

$$\Longrightarrow \quad |\psi_i\rangle = e^{\frac{iq\Lambda_{ij}}{\hbar c}}\,|\psi_j\rangle$$



relate i to j using gauge transformation w/ function Λ_{ij}

Can rewrite $\vec{\nabla} \Lambda_{ij} = \frac{\hbar c}{iq} \, e^{-\frac{iq\Lambda_{ij}}{\hbar c}} \, \vec{\nabla} \, e^{\frac{iq\Lambda_{ij}}{\hbar c}}$

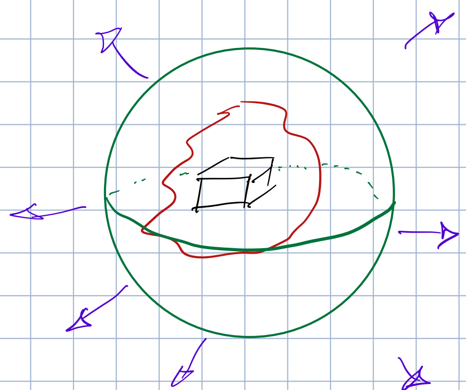
$$g(\Lambda_{ij}) = e^{\frac{iq\Lambda_{ij}}{\hbar c}} \qquad g^{-1}(\Lambda_{ij}) = e^{-\frac{iq\Lambda_{ij}}{\hbar c}}$$

$$\Psi_i = g(\Lambda_{ij}) \, \Psi_j$$

$$A_i = g^{-1}(\Lambda_{ij}) \left(\vec{A}_j - \frac{i\hbar c}{q} \vec{\nabla} \right) g(\Lambda_{ij})$$

$$g(\Lambda_1) \, g(\Lambda_2) = g(\Lambda_1 + \Lambda_2)$$
$$g(\Lambda) \, g^{-1}(\Lambda) = \mathbb{1}$$

gauge group of transformation

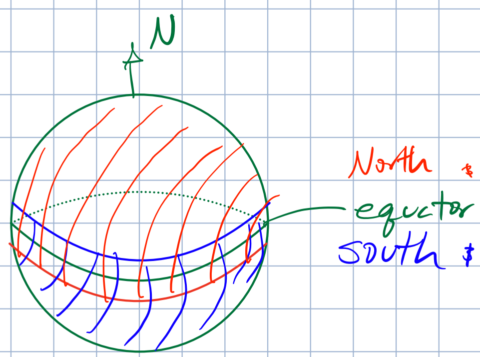


idk whats inside

can only describe $\vec{B}$ outside

specifically, some sphere outside

$$\vec{B} = \frac{g}{r^2} \hat{e}_r$$



North & lil south U_N

equator

south & lil north U_S

define $\vec{A}_N$ & Ψ_N
& $\vec{A}_S$ & Ψ_S

(r, θ, φ)

on North $\quad 0 \le \theta \le \pi/2 + \epsilon$

$$\vec{A}_N = \frac{g}{r} \, \frac{1-\cos\theta}{\sin\theta} \, \hat{e}_\varphi$$

$$\vec{B}_N = \vec{\nabla} \times \vec{A}_N = \frac{\hat{e}_r}{r\sin\theta} \left(\frac{\partial}{\partial\theta}(\sin\theta A_\varphi) - \frac{\partial A_\theta}{\partial\varphi} \right) + \cancelto{0}{\hat{e}_\theta [\ ]} + \cancelto{0}{\hat{e}_\varphi [\ ]}$$

$$= \frac{g}{r^2} \hat{e}_r$$

$\vec{A}_N$ is well defined on N as $\theta \to 0$ $\qquad \frac{1-\cos\theta}{\sin\theta} \sim \theta \sim 0$

but $\vec{A}_N$ is singular @ $\theta = \pi$

on South

$$\vec{A}_S = -\frac{g}{r} \frac{1+\cos\theta}{\sin\theta} \hat{e}_\varphi \qquad \text{well defined @ } \theta = \pi$$

compute $\vec{B}_S = \vec{\nabla} \times \vec{A}_S = \frac{g}{r^2} \hat{e}_r$

so $\vec{B} = \frac{g}{r^2} \hat{e}_r$ in N & S

overlap region around equator

$$\vec{A}_N = g^{-1}(\Lambda_{NS}) \left(\vec{A}_S - \frac{i\hbar c}{q} \vec{\nabla} \right) g(\Lambda_{NS})$$

@ $\theta = \pi/2$, $\quad \vec{A}_N = \frac{g}{r} \hat{e}_\varphi$
$\qquad\qquad\quad \vec{A}_S = -\frac{g}{r} \hat{e}_\varphi$

$\vec{A}_N = \vec{A}_S + \vec{\nabla} \Lambda_{NS}$ w/ $\Lambda_{NS} = 2g\varphi$

$g(\Lambda_{NS}) = e^{iq(2g\varphi)/\hbar c}$

Is g_{NS} a well defined f^n along equator?

$g(\Lambda_{NS})$ is f^n of φ

$g(\Lambda_{NS}, \varphi) = g(\Lambda_{NS}, \varphi + 2\pi)$ $\qquad$ b/c $g(\Lambda_{NS})$ is periodic f^n

$$\Longrightarrow \frac{q2g}{\hbar c} 2\cancel{\pi} = \cancel{2\pi} n \qquad n \in \mathbb{Z}$$

$$\frac{n}{2} = \frac{qg}{\hbar c} \qquad \text{Dirac quantization condition}$$

① g will be $\frac{\hbar c}{q} \cdot \frac{n}{2} = |g|$ for some n

② if there's 1 magnetic monopole anywhere, by consistency w/ QM

$$\Longrightarrow q = \frac{n}{2} \frac{\hbar c}{g} \qquad \text{where } g \text{ is strength of magnetic monopole}$$

$\longrightarrow$ predicts quantization of electric charge

lower limit of g? $\qquad q \cdot g = \frac{n}{2} \cdot \hbar c$ $\qquad$ "principle U(1) fibre bundle"

n - winding # of wrap around S^1 $\qquad$ 0 1 2 $\qquad$ n = value of first Chern class of this bundle