$L^2 \; Y_{lm}(\theta,\phi) = l(l+1)\hbar^2 \; Y_{lm}(\theta,\phi)$
$L_z \; Y_{lm}(\theta,\phi) = m\hbar \; Y_{lm}(\theta,\phi)$

$Y_{lm}(\theta,\phi) = \langle \theta,\phi | l,m \rangle$

SHO $\quad |n\rangle$
$\psi_n(x) = \langle x|n\rangle$
$a|0\rangle = 0 \Leftarrow l \; \psi_0(x)$
$|n\rangle \propto (a^\dagger)^n |0\rangle$

$L_z = -i\hbar \frac{\partial}{\partial \phi}$

$-i\hbar \frac{\partial}{\partial \phi} Y_{lm}(\theta,\phi) = m\hbar \; Y_{lm}(\theta,\phi)$

soln: $\quad Y_{lm}(\theta,\phi) = e^{im\phi} \; \Theta_{lm}(\theta)$

$L_+ |l,l\rangle = 0 \qquad L_- |l,-l\rangle = 0$

$L_+ = L_x + iL_y = (x+iy)p_z + z(-p_y + ip_x)$

$L_+ \; Y_{ll}(\theta,\phi) = 0$

$L_+ = \hbar e^{i\phi} \left(\frac{\partial}{\partial\theta} + i\cot\theta \frac{\partial}{\partial\phi} \right)$

$\hbar e^{i\phi} \left(\frac{\partial}{\partial\theta} + i\cot\theta \frac{\partial}{\partial\phi} \right) e^{il\phi} \; \Theta_{ll}(\theta) = 0$

$\hbar e^{i\phi} \left(\frac{\partial}{\partial\theta} - l\cot\theta \right) e^{i\phi} \; \Theta_{ll}(\theta) = 0$

$\left(\frac{\partial}{\partial\theta} - l\cot\theta \right) \Theta_{ll}(\theta) = 0$

$\frac{d\Theta}{d\theta} = l \frac{\cos\theta}{\sin\theta} \Theta(\theta)$

$\int \frac{d\Theta_{ll}(\theta)}{\Theta_{ll}(\theta)} = l \int \frac{\cos\theta}{\sin\theta} d\theta$

$\ln|\Theta_{ll}(\theta)| = l \cdot \ln|\sin\theta| + C$

$\Theta_{ll}(\theta) = C \cdot (\sin\theta)^l$

$L_- = -\hbar e^{-i\phi} \left(\frac{\partial}{\partial\theta} - i\cot \frac{\partial}{\partial\phi} \right)$

$Y(\theta,\phi) = C_{lm} \; (L_-)^{l-m} \; e^{il\phi} (\sin\theta)^l$

in 3D: $\quad \int d^3r \; |\Psi(\vec{r})|^2 = 1 \qquad$ for $\Psi(\vec{r})$

$$\Phi(\vec{r}) = R_{Elm}(r)\, Y_{lm}(\theta,\phi)$$

$$\underbrace{\int_0^{\infty} r^2 \left| R_{Elm}(r)\right|^2}_{1} \quad \underbrace{\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \quad \left| Y_{lm}(\theta,\phi)\right|^2}_{\int d\Omega\, \left|Y_{lm}(\theta,\phi)\right|^2 = 1}$$

fixes $|C_{lm}|^2 = 1$

$$Y_{00} = \frac{1}{\sqrt{4\pi}}$$

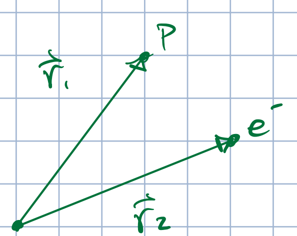
$$Y_{11} = -\sqrt{\frac{3}{8\pi}}\, e^{i\phi} \sin\theta$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$Y_{1,-1} = \sqrt{\frac{3}{8\pi}}\, e^{-i\phi} \sin\theta$$

$l = 1/2,\ 3/2, \ldots$ not allowed for $Y_{lm}(\theta,\phi)$

Hydrogen Atom



$$H = \frac{\vec{p}_1^{\,2}}{2m_p} + \frac{\vec{p}_2^{\,2}}{2m_e} + V(\vec{r}_1 - \vec{r}_2) \qquad V(\vec{r}) = \frac{-e^2}{r}$$

e – charge of electron

$$\vec{R} = \frac{m_p \vec{r}_1}{M} + \frac{m_e \vec{r}_2}{M}$$

$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

$$M = m_p + m_e$$

$$\mu = \frac{m_p m_e}{M}$$

$$H_e = \frac{\vec{P}^2}{2M} + \frac{\vec{p}^{\,2}}{2\mu} + V(r) = \frac{\vec{P}^2}{2M} + H_{red.} \qquad r = |\vec{r}|$$

$\vec{p}$ – relative momentum
$\vec{P}$ – C.O.M. momentum

$$\vec{P} = -i\hbar \vec{\nabla}_R$$

$$\phi(\vec{r}, \vec{R}) = e^{\vec{P}\cdot\vec{R}/\hbar} \, \phi(\vec{r})$$

$$\vec{p} = -i\hbar \vec{\nabla}_r$$

$$H_{rel} \, \phi(\vec{r}) = E_{rel} \, \phi(\vec{r})$$

$$E = \frac{\vec{P}^2}{2M} + E_{rel}$$

$$\left(-\frac{\hbar^2}{2\mu} \vec{\nabla}_r^{\,2} + V(r)\right) \phi_{E_{rel}} = E_{rel} \, \phi_{E_{rel}}(\vec{r})$$

$$[H_{red}, L^2] = [H_{red}, L_z] = [L^2, L_z]$$

$$\left[-\frac{\hbar^2}{2\mu}\left(\frac{1}{r^2}\left(r\frac{\partial}{\partial r}\right)\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial r} - \frac{\overset{\curvearrowright\, -\frac{l(l+1)}{r^2}}{L^2}}{\hbar^2 r^2}\right) + V(r)\right] \phi_E(\vec{r})$$

$$\phi_E(\vec{r}) = R_{El}(r) \cdot Y_{lm}(\theta, \phi) \qquad L^2 \phi_E(\vec{r}) = l(l+1)\hbar^2 \phi_E(\vec{r})$$

energy eigenvalue will appear

E not a fn of m

$$\left[-\frac{\hbar^2}{2\mu}\left(\frac{1}{r^2}\left(r\frac{\partial}{\partial r}\right)\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial r} - \frac{l(l+1)}{r^2}\right) + V(r)\right] R_{El}(r) = E \, R_{El}(r)$$

potential b/c l appears here