$\langle\psi|\psi\rangle \geq 0$

$\mathcal{O}|\psi\rangle \rightarrow \langle\psi|\mathcal{O}^+\mathcal{O}|\psi\rangle \geq 0$

$L_i^+ = L_i \quad , \quad L_\pm^+ = L_\mp$

$\frac{1}{2}(L_-L_+ + L_+L_-) = \frac{1}{2}(L_+^+ L_+ + L_-^+ L_-) = L_x^2 + L_y^2 = L - L_z^2$

$\langle\alpha,\beta|\, L^2 - L_z^2 \,|\alpha,\beta\rangle = (\alpha - \beta^2)\, \langle\alpha\beta|\alpha\beta\rangle$

$= \langle\alpha\beta|\, \frac{1}{2} L_+^+ L_+ + \frac{1}{2} L_-^+ L_-)\, |\alpha,\beta\rangle$

$= \frac{1}{2}\langle\alpha\beta|\, L_+^+ L_+ \,|\alpha\beta\rangle + \frac{1}{2}\langle\alpha\beta|\, L_-^+ L_- \,|\alpha,\beta\rangle \geq 0$

$\rightsquigarrow \alpha - \beta^2 \geq 0$

$|\alpha, \beta+\hbar\rangle$

$|\alpha, \beta\rangle$

$|\alpha, \beta-\hbar\rangle$

stop when $L_\pm|\alpha,\beta\rangle = 0$

Some β_{max}, β_{min}, and α_{min}

$L_+|\alpha, \beta_{max}\rangle = 0$

$L_-|\alpha, \beta_{min}\rangle = 0$

$|\alpha, \beta_{max}\rangle$

$|\alpha, \beta_{max}-\hbar\rangle$

$\vdots$

$|\alpha, \beta\rangle$

$\vdots$

$|\alpha, \beta_{min}+\hbar\rangle$

$|\alpha, \beta_{min}\rangle$

$$L_-L_+ = (L_x - iL_y)(L_x + iL_y) = L_x^2 + L_y^2 + (iL_xL_y - iL_yL_x)$$

$$= L_x^2 + L_y^2 + i[L_x, L_y]$$

$$= L^2 - L_z^2 + i(i\hbar L_z)$$

$$= L^2 - L_z^2 - \hbar L_z$$

" technology keeps killing itself "

$$L_+L_- = L_x^2 + L_y^2 + \hbar L_z = L^2 - L_z^2 + \hbar L_z$$

$$L_-L_+|\alpha,\beta_{max}\rangle = 0 = (L^2 - L_z^2 - \hbar L_z)|\alpha, \beta_{max}\rangle$$

$$= (\alpha - \beta_{max}^2 - \hbar\beta_{max})\,|\alpha, \beta_{max}\rangle$$

$$\rightarrow \alpha = \beta_{max}(\beta_{max} + \hbar)$$

$L_+ L_- |\alpha, \beta_{min}\rangle = 0$

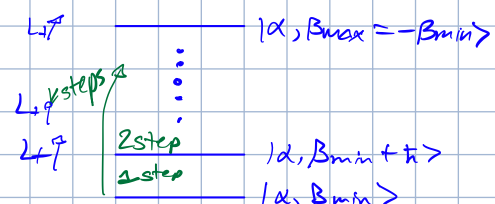
$\rightarrow \alpha = \beta_{min}(\beta_{min} - \hbar)$

$\beta_{max} = -\beta_{min} \quad (\beta_{min} < 0)$

$L_- L_+ |\alpha, \beta_{max}\rangle$

$\beta_{min}(\beta_{min} - \hbar) = \beta_{max}(\beta_{max} + \hbar)$

can look like quadratic β_{max} eq^n



$\beta_{max} - \beta_{min} = k\hbar \qquad k$ # of steps: 0, 1, ..

$\beta_{max} - (-\beta_{max}) = 2\beta_{max} = k\hbar$

$\beta_{max} = k\frac{\hbar}{2}$

$\alpha = \hbar^2 \frac{k}{2}\left(\frac{k}{2}+1\right) = k\frac{\hbar}{2}\left(k\frac{\hbar}{2} + \hbar\right) = \hbar^2 \frac{k}{2}\left(\frac{k}{2}+1\right)$

When $k=0$,

$\alpha = 0 \qquad \beta_{max} = -\beta_{min} = 0$

$|0,0\rangle \qquad L^2|0,0\rangle = 0$

$L_z|0,0\rangle = 0$

spherically symmetric. generators of rotation do nothing

$k=1$

$\alpha = \hbar^2 \frac{1}{2}\left(1+\frac{1}{2}\right) = \frac{3}{4}\hbar^2$

$\beta_{min} = -\hbar/2 \qquad \beta_{max} = +\hbar/2$

2 states: $|\frac{3}{4}\hbar^2, \frac{\hbar}{2}\rangle$, $|\frac{3}{4}\hbar^2, -\frac{\hbar}{2}\rangle$

New notation: $L^2|l,m\rangle = l(l+1)|l,m\rangle \qquad l = 0, \frac{1}{2}, 1, \ldots$

$L_z|l,m\rangle = m\hbar\,|l,m\rangle$

$L^2|\alpha,\beta\rangle = \alpha|\alpha,\beta\rangle$

$L_z|\alpha,\beta\rangle = \beta|\alpha,\beta\rangle$

$L_+|\frac{1}{2}, \frac{1}{2}\rangle = 0$

$L_-|\frac{1}{2}, -\frac{1}{2}\rangle = 0$

$L_+|\frac{1}{2}, -\frac{1}{2}\rangle \propto |\frac{1}{2}, \frac{1}{2}\rangle$

$L_-|\frac{1}{2}, \frac{1}{2}\rangle \propto |\frac{1}{2}, -\frac{1}{2}\rangle$

$\langle 1/2, 1/2 | 1/2, 1/2 \rangle = 0$

$\langle 1/2, 1/2 | 1/2, 1/2 \rangle = 1 = \langle 1/2, -1/2 | 1/2, -1/2 \rangle$

Want to find 3 2x2 Matrices

$$\begin{pmatrix} \langle 1/2, 1/2 | L_x | 1/2, 1/2 \rangle & \langle 1/2, 1/2 | L_x | 1/2, -1/2 \rangle \\ \langle 1/2, -1/2 | L_x | 1/2, 1/2 \rangle & \langle 1/2, -1/2 | L_x | 1/2, -1/2 \rangle \end{pmatrix} = \text{"}L_x\text{"}$$

same for L_y & L_z

$$L_z = \begin{pmatrix} \hbar/2 & 0 \\ 0 & \hbar/2 \end{pmatrix}$$

$$L_x = \frac{1}{2}(L_+ + L_-)$$
$$L_y = -\frac{i}{2}(L_+ - L_-)$$

just find L_+ & L_- matrix elements

$L_- | 1/2, 1/2 \rangle = c \, | 1/2, -1/2 \rangle$ $\rightarrow$ $\langle \frac{1}{2}, \frac{1}{2} | L_+ = \langle \frac{1}{2}, -\frac{1}{2} | c^*$

take top rung & lower

$L_- | 1/2, -1/2 \rangle = 0$

lower bottom rung

$L_+ | 1/2, 1/2 \rangle = 0$

raise top rung

$L_+ | 1/2, -1/2 \rangle = d \, | 1/2, 1/2 \rangle$ $\rightarrow$ $\langle 1/2, -1/2 | L_- = \langle 1/2, 1/2 | d^*$

raise low rung

$$1 = \langle 1/2, -1/2 | 1/2, -1/2 \rangle = \frac{1}{|c|^2} \langle 1/2, 1/2 | L_+ L_- | 1/2, 1/2 \rangle$$
$$= \frac{1}{|c|^2} \langle 1/2, 1/2 | L^2 - L_z^2 + \hbar L_z | 1/2, 1/2 \rangle$$
$$= \frac{1}{|c|^2} \left(\tfrac{1}{2}(\tfrac{1}{2}+1) - (\tfrac{1}{2})^2 + (\tfrac{1}{2}) \right) \hbar^2 = \frac{\hbar^2}{|c|^2} \rightarrow c = \hbar$$

$$\text{"}L_-\text{"} = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\text{"}L_+\text{"} = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\text{"}L_z\text{"} = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{"}L_x\text{"} = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{"}L_y\text{"} = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$(\text{"}L_x\text{"}, \text{"}L_y\text{"}, \text{"}L_z\text{"}) = \frac{\hbar}{2}(\sigma_x, \sigma_y, \sigma_z)$$

$$S_i = \frac{\hbar}{2} \sigma_i$$

spin operators, for spin $\frac{1}{2}$