$$H = \frac{\vec{p}^{\,2}}{2m} + V(r)$$

$$[H, L_z] = 0$$

$$[H, \vec{L}\cdot\vec{L}] = 0$$

$$[\vec{L}\cdot\vec{L}, L_z] = 0$$

Write H in terms of r, $\frac{\partial}{\partial r}$, + L^2

↑ all angular $\frac{\partial}{\partial \phi}$ $\frac{\partial}{\partial \theta}$

Show allowed $\vec{L}^2$, L_z eigenvalues are

$\vec{L}^2 |l, m\rangle$ $\qquad l = 0, 1, 2, \dots, \infty$

$\qquad\qquad m = -l, -l+1, \dots, l$

$$\vec{L}^2 \, |l, m\rangle = l(l+1)\hbar \, |l, m\rangle$$
$$L_z \, |l, m\rangle = m \hbar \, |l, m\rangle$$

$$\left.\begin{array}{l} |l, l\rangle \\ |l, l-1\rangle \\ |l, l-2\rangle \\ \vdots \\ |l, -l\rangle \end{array}\right\} \text{2l+1 sides for fixed } l$$

Damn never thought listening in HS chem would be helpful

$$\langle \theta \phi, l m\rangle = Y_{lm}(\theta, \phi)$$

Use all this to solve H-atom

$\vec{L}^2$, L^2, $\vec{L}\cdot\vec{L}$, $L_i L_i$

operators

$$\vec{L}\cdot\vec{L} = L_{ii} \qquad L_i = \epsilon_{ijk} r_j p_k$$

$$L_i L_i = \epsilon_{ijk} r_j p_k \, \epsilon_{imn} r_m p_n$$

$$= (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) r_j p_k r_m p_n$$

$$= r_j p_n r_j p_n - r_n p_k r_k p_n$$

$$\vec p^{\,2}, \quad \vec r\cdot\vec p = -i\hbar\,\vec r\,\vec\nabla$$

$$[p_n, r_j] = p_n r_j - r_j p_n = -i\hbar\delta_{nj}$$

$$p_n r_j = r_j p_n - i\hbar\delta_{nj}$$

$$r_j p_n = p_n r_j + i\hbar\delta_{nj}$$

$$= r_j(r_j p_n - i\hbar\delta_{jn})p_n - r_n p_k(p_n r_k + i\hbar\delta_{nj})$$

$$= r_j r_j p_n p_n - i\hbar r_j p_n \delta_{jn} - r_n p_k p_n r_k - i\hbar r_n p_k \delta_{nj}$$

$$[p_k, p_n] = 0$$

$$= \vec r^{\,2}\cdot\vec p^{\,2} - i\hbar\,\vec r\cdot\vec p - r_n p_n p_k r_k - i\hbar\,\vec r\cdot\vec p$$

$$r_n p_n(r_k p_k - i\hbar\underbrace{\delta_{kk}}_{\to 3})$$

$$r_n p_n(r_k p_k - 3i\hbar)$$

$$r_n p_n r_k p_k - 3i\hbar\, r_n p_n$$

$$= \vec r^{\,2}\cdot\vec p^{\,2} - i\hbar\,\vec r\cdot\vec p - \vec r\cdot\vec p\ \vec r\cdot\vec p + 3i\hbar\,\vec r\cdot\vec p - i\hbar\,\vec r\cdot\vec p$$

$$= \vec r^{\,2}\cdot\vec p^{\,2} - (\vec r\cdot\vec p)^2 + i\hbar\,\vec r\cdot\vec p = \vec L^2$$

$$\vec p^{\,2} - \frac{1}{r^2}\left(-i\hbar r\frac{\partial}{\partial r}\right)^2 + i\hbar\frac{1}{r^2}\left(-i\hbar r\frac{\partial}{\partial r}\right) = \frac{L^2}{r^2}$$

$$\vec p = -i\hbar\vec\nabla$$
$$\vec\nabla = \hat e_r\frac{\partial}{\partial r} + \hat e_\theta(\ ) + \hat e_\phi(\ )$$
$$\vec r = r\hat e_r$$

$$\vec r\cdot\vec p = -i\hbar r\frac{\partial}{\partial r}$$

$$\vec p^{\,2} = \frac{1}{r^2}\left(L^2 - \hbar^2 r\frac{\partial}{\partial r} - \hbar^2\left(r\frac{\partial}{\partial r}\right)\left(r\frac{\partial}{\partial r}\right)\right)$$

$$H = -\frac{\hbar^2}{2m}\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{L^2}{\hbar^2 r^2}\right) + V(r)$$

$\to$ radial ODE $\quad H(\phi) = E(\phi) \quad$ after we find $\vec L^2$ eigenstates

$$\phi(r,\theta,\phi) = R_\ell(r)\,Y_{\ell m}(\theta,\phi)$$

$$\vec L^2 Y_{\ell,m} = \hbar^2\ell(\ell+1)\,Y_{\ell m}(\theta,\phi)$$

Two methods

coordinate space methods

operator methods

$[L_i, L_j] = i\hbar \epsilon_{ijk} L_k$

$L_\pm = L_x \pm i L_y$

$[L_z, L_\pm] = [L_z, L_x \pm i L_y] = i\hbar L_y \pm i(-i\hbar L_x)$
$= \pm \hbar L_\pm$

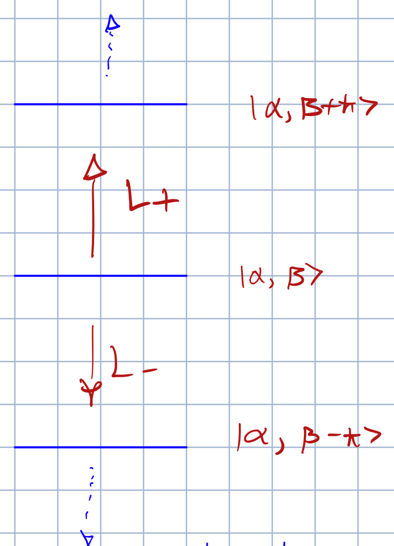
$[L_z, L_\pm] = \pm\hbar L_\pm$ SHO : $[H, a^+] = \hbar\omega a^+$ $[H, a] = -\hbar\omega a$

$L^2 |\alpha, \beta\rangle = \alpha |\alpha, \beta\rangle$
$L_z |\alpha, \beta\rangle = \beta |\alpha, \beta\rangle$

$[L^2, L_\pm] = 0 \rightarrow L_\pm |\alpha, \beta\rangle$ has same L^2 eigenvalue

$L^2 L_\pm |\alpha, \beta\rangle = L_\pm L^2 |\alpha, \beta\rangle$
$= L_\pm \alpha |\alpha, \beta\rangle = \alpha L_\pm |\alpha, \beta\rangle$

$L_z L_\pm |\alpha, \beta\rangle = (L_\pm L_z \pm \hbar L_\pm) |\alpha, \beta\rangle = (\beta \pm \hbar) |\alpha, \beta\rangle$



terminates @ both ends
b/c can't have negative norm