"But in QM, things are more complicated"

Rotations in 3-D & implementation in QM

① Group elements depend on 3 parameters

② Rotation group is nonabelian

Translations: $\phi(x) \to \phi(x-\epsilon) = T(\epsilon)\,\phi(x) = e^{-i\epsilon \hat{p}/\hbar}\,\phi(x)$

Hermitian

parameter

$\psi(\vec{x},t) \in \mathcal{H}$

$|\psi(x)\rangle \in \mathcal{H}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \dashrightarrow \overset{3\times 3}{R} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$x^2+y^2+z^2$ - invariant $\to R^T R = \mathbb{1}$

$\phi(\vec{r}) \longrightarrow \phi(\vec{r}\,')$

specify rotation by axis $\hat{n}$ (leave this direction fixed) & an angle ϵ

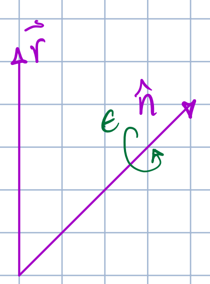
$\phi(\vec{r}\,') = e^{-i\epsilon \hat{n}\hat{L}}\,\phi(\vec{r})$

where $\vec{L} = \vec{r} \times \vec{p}$

$\hat{L} = \hat{\vec{r}} \times \hat{\vec{p}}$

3 generators $L_x \quad L_y \quad L_z$

$L_i = \sum_j \sum_k \epsilon_{ijk}\, r_j\, p_k$



$\vec{r} \to \vec{r} - \epsilon\, \hat{n} \times \vec{r} \quad + \mathcal{O}(\epsilon^2)$

$\vec{r} = r_n \hat{n} + \vec{r}_\perp \qquad \vec{r}_\perp \cdot \hat{n} = 0$

$\hat{n} = \hat{e}_z$

$\vec{r} \to \vec{r} - \epsilon \hat{e}_z \times \vec{r} = \vec{r} - \epsilon\, \hat{e}_z \times (x\hat{e}_x + y\hat{e}_y + z\hat{e}_z)$

$x \longrightarrow x + \epsilon y \qquad z \to z$

$y \longrightarrow y - \epsilon x$

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos & \sin \\ -\sin & \cos \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \underset{\text{small } \epsilon}{\approx} \begin{pmatrix} 1 & \epsilon \\ -\epsilon & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + \epsilon y \\ y - \epsilon x \end{pmatrix}$$

Want to show for $\hat{n} = \hat{e}_z$

$$\phi(x', y', z') = e^{-i\epsilon L_z/\hbar} \cdot \phi(x,y,z) \qquad \text{to order } \epsilon$$

L_z: generator of rotations about $\hat{e}_z$

$$\phi(x+\epsilon y, y-\epsilon x, z) = \phi(x,y,z) + \epsilon y \frac{\partial \phi}{\partial x} - \epsilon x \frac{\partial \phi}{\partial y} + \mathcal{O}(\epsilon^2) \qquad \text{Taylor series}$$

Right hand side

$$e^{-i\epsilon L_z/\hbar} \phi(x,y,z) = e^{-i\epsilon(xp_y - yp_x)/\hbar} \phi(x,y,z)$$

$$= \left(1 - \tfrac{i}{\hbar}\epsilon(xp_y - yp_x) + \mathcal{O}(\epsilon^2)\right) \phi(x,y,z)$$

$$= \left(1 + \epsilon y \frac{\partial}{\partial x} - \epsilon x \frac{\partial}{\partial y} + \mathcal{O}(\epsilon^2)\right) \phi(x,y,z)$$

$$[L_x, L_y] = [yp_z - zp_y,\ zp_x - xp_z]$$

$$= yp_x [p_z, z] + p_y x [z, p_z]$$

$$= -i\hbar (yp_x - p_y x) = i\hbar (xp_y - yp_x) = i\hbar L_z$$

$$L_i = (L_1, L_2, L_3)$$

$$[L_i, L_j] = i\hbar\, \epsilon_{ijk} L_k$$

$$\epsilon_{ijk} \begin{cases} 0 & \text{any two or three } i=j,\ j=k,\ i=k \\ 1 & ijk = 123 \quad 231 \quad 312 \\ -1 & ijk = 132 \quad 321 \quad 132 \end{cases}$$

$$\epsilon_{ijk}\, \epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$$

$$(\vec{b} \times \vec{c})_i = \epsilon_{ijk}\, b_j\, c_k$$

$$(\vec{a} \times (\vec{b} \times \vec{c}))_i = \epsilon_{ijk}\, a_j \cdot (\vec{b} \times \vec{c})_k = \epsilon_{ijk}\, a_j\, (\epsilon_{kmn}\, b_m\, c_n)$$

$$= \epsilon_{kij}\, a_j\, \epsilon_{kmn}\, b_m\, c_n$$

$$= \epsilon_{kij}\, \epsilon_{kmn}\, a_j\, b_m\, c_n$$

$$= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})\, a_j\, b_m\, c_n$$

$$= b_i\, \vec{a} \cdot \vec{c} - c_i\, \vec{a} \cdot \vec{b}$$

$$[H, L_x] = [H, L_y] = [H, L_z] = 0$$

$$[L_x, L_y] = i\hbar L_z$$

Can only choose one L_i to find energy eigenstates for