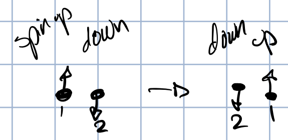
Spin statistics Th$^{\underline{m}}$

there are identical particles in nature, can't tell them apart
operator P_{12} exchanges 2 identical particles: P_{12} :



$[H, P_{12}] = 0$ $\quad P_{12}^2 = 1 \Rightarrow P_{12}$ has eigenvalues ± 1

⓪ All particles in 3+1 dims. can be divided into 2 groups

Fermions - spin $1/2$, $3/2$, $5/2$, ...
e^- quarks p n ν

Bosons - spin 0, 1, ...
photon γ Higgs nuclei

identical fermions - in $P_{12} = -1$ eigenstates
identical bosons - in $P_{12} = +1$ eigenstates

example 1: Two identical bosons (spin=0) in external potential (not interacting w/ each other)

$$H = \frac{\vec{p}_1^{\,2}}{2m} + \frac{\vec{p}_2^{\,2}}{2m} + V(\vec{r}_1) + V(\vec{r}_2)$$

$$= H_1 + H_2$$

$\vec{p}_1 = -i\hbar \vec{\nabla}_{r_1}$
$\vec{p}_2 = -i\hbar \vec{\nabla}_{r_2}$

let $\psi_a(\vec{r}_1)$ be energy eigenstate of H_1

includes all quantum #'s to label state (n, l, m) for Hydrogen
(n_x, n_y, n_z) for 3D SHO

Wave f$^{\underline{n}}$ $\psi(\vec{r}_1, \vec{r}_2) \in L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3)$

$P_{12} : \psi(\vec{r}_1, \vec{r}_2) \to \psi(\vec{r}_2, \vec{r}_1)$

energy eigenstates: $\psi_{ab}(\vec{r}_1, \vec{r}_2) = \psi_a(\vec{r}_1) \otimes \psi_b(\vec{r}_2)$ change slots

$H \psi_{ab}(\vec{r}_1, \vec{r}_2) = (E_a + E_b)\, \psi_{ab}(\vec{r}_1, \vec{r}_2)$

we want to impose Bose statistics
$\to$ make this eigenstate of P_{12}

$P_{12} \psi^{\pm}_{ab}(\vec{r}_1, \vec{r}_2) = \pm \psi^{\pm}_{ab}(\vec{r}_1, \vec{r}_2)$

particle 1 in state a 2 in state a
... states ... in state b

$$\psi_{ab}^{\pm}(\vec r_1,\vec r_2) = c\left(\psi_a(\vec r_1)\otimes\psi_b(\vec r_2) \pm \psi_a(\vec r_2)\otimes\psi_b(\vec r_1)\right)$$

$\psi_{ab}^{+}(\vec r_1,\vec r_2)$ is only allowed wave f^{n} for identical bosons
fermions use ψ_{ab}^{-}

assume $\int d^3r\, |\psi_a(\vec r)|^2 = 1$

demand $\int d^3r_1 \int d^3 r_2\, |\psi^{+}(\vec r_1,\vec r_2)|^2 = 1$

$$\to = |c|^2 \int d^3r_1 \int d^3r_2 \left(\psi_a^*(\vec r_1)\psi_b^*(\vec r_2) + \psi_a^*(\vec r_2)\psi_b^*(\vec r_1)\right)$$
$$\times\left(\psi_a(\vec r_1)\psi_b(\vec r_2) + \psi_a(\vec r_2)\psi_b(\vec r_1)\right)$$

$a\neq b$ | $a=b$

$\psi_a^*(\vec r_1)\,\psi_b^*(\vec r_2)\,\psi_a(\vec r_1)\,\psi_b(\vec r_2)$

δ_{ab}

if $a\neq b$ $\quad \int \psi_a^*(r_1)\psi_b(r_1)\, d^3r_1 = 0$ $\quad + \cancel{\psi_a^*(\vec r_1)}\,\psi_b^*(\vec r_2)\,\cancel{\psi_a(\vec r_2)}\,\cancel{\psi_b(\vec r_1)}$

$1 = |c|^2(1+1+1+1)$
$\to \frac{1}{4} = |c|^2$
$c = \frac{1}{\sqrt4} = \frac12$

if $a\neq b$ $\quad c = \frac{1}{\sqrt2}$
if $a=b$ $\quad c = \frac12$

$+ \cancel{\psi_a^*(\vec r_2)}\,\cancel{\psi_b^*(\vec r_1)}\,\cancel{\psi_a(\vec r_1)}\,\psi_b(\vec r_2)$

$+ \psi_a^*(\vec r_2)\,\psi_b^*(\vec r_1)\,\psi_a(\vec r_2)\,\psi_b(\vec r_1)$

$1 = |c|^2(1+1) \to |c|^2 = \frac12 \to c = \frac{1}{\sqrt2}$

identical spin 1/2 fermions

$$\psi \in \left(\underbrace{L^2(\mathbb{R}^3)}_{\text{spatial of 1st}}\otimes\underbrace{\mathbb{C}^2}_{\text{spin of 1st}}\right)\otimes\left(\underbrace{L^2(\mathbb{R}^3)}_{\text{spatial 2nd}}\otimes\underbrace{\mathbb{C}^2}_{\text{spin 2nd}}\right)$$

$P_{12}\quad \underset{\pm}{|\alpha\rangle_1} \otimes \underset{\pm}{|\beta\rangle_2} \to |\beta\rangle_1\otimes|\alpha\rangle_2$

$\vec S = \vec S_1 + \vec S_2$

$|s=1, m\rangle \quad m = 1,0,-1 \quad P_{12} = +1$

$|+\rangle_1\otimes|+\rangle_2$

$\frac{1}{\sqrt2}\left(|+\rangle_1\otimes|-\rangle_1 + |-\rangle_1\otimes|+\rangle\right)$

$|-\rangle_1\otimes|-\rangle_2$

→ symmetric

$|s=0, m=0\rangle$

$\frac{1}{\sqrt2}\left(|+\rangle_1\otimes|-\rangle_2 - |-\rangle_1\otimes|+\rangle_2\right)$

$P_{12} = -1$

→ antisymmetric

2 possible wave f^{n}'s obeying Fermi

$\underbrace{\psi_{ab}^{+}(\vec r_1,\vec r_2)}_{\text{space sym}} \otimes \underbrace{|s=0, m=0\rangle}_{\text{spin antisym}}$ $\qquad P_{12} = \underbrace{(+1)}_{\text{space}}\underbrace{(-1)}_{\text{spin}} = -1$

$\psi_{ab}^{-}(\vec{r}_1, \vec{r}_2) \otimes |s=1, m\rangle$ $\quad P_{12} = (-1)(+1) = -1$

space antisym $\quad$ spin sym $\quad$ space spin

Fermi statistics: total wave fk (incl. space & spin) be antisymmetric