H atom

$H_{rel}\, \phi(\vec{r}) = E\, \phi(\vec{r})$

rel. coord $\vec{r}$

$p^2$ - relate to $\frac{\partial}{\partial r}$ $L^2$

$\phi(\vec{r}) = R_{El}(r)\, Y_{lm}(\theta, \phi)$ $\rightsquigarrow$ horrible eq$^{n}$ for $R_{el}$

Substitution $R_{el}(r) = \frac{U_{el}(r)}{r}$

$$r \frac{d}{dr}\left(R_{el}(r)\right) = r \frac{d}{dr}\left(\frac{U_{el}(r)}{r}\right) = \frac{dU_{el}}{dr} - \frac{U_{el}}{r} + \dots$$

$$\left[-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + V(r) + \frac{l(l+1)\hbar^2}{2\mu r^2}\right] U_{el}(r) = E\, U_{el}(r)$$

looks like 1-D SE but

① $r \in (0, \infty)$

② analog of 1-D potential is $V_{effective}(r) = V(r) + \frac{l(l+1)\hbar^2}{2\mu r^2}$

larger L, larger barrier

repulsive potential

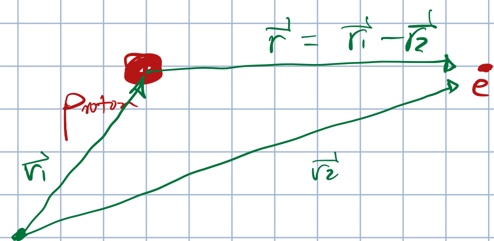
③ Boundary conditions

$r \to \infty$

$\int_0^\infty |R_{El}(r)|\, r^2 dr = \text{finite } \#$

$= \int_0^\infty |U_{El}(r)|^2 dr$

$U_{el} \to 0$ as $r \to \infty$




$r \to 0$

$U_{El}(r=0) = ???\ \equiv 0$

defined as so

$d^3x = r^3 \sin\theta\, dr\, d\theta\, d\phi$

Choose $V = \frac{-e^2}{r}$

Asymptotic Analysis as $r \to \infty$

$$-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} U_{El}(r) \sim E\, U_{El}(r)$$

$E < 0$ for normalizable bound state (some damping)

$$U_{El} \sim e^{-Kr}$$

$$K^2 = -\frac{2\mu E}{\hbar^2}$$ kinda like $k^2$, $q^2$

define $\rho = K \cdot r$

$$U_{El} \underset{\rho \to \infty}{\sim} e^{-\rho}$$

to describe all $\rho$, choose eq$^{k}$ like last quarter: $U_{El}(\rho) = e^{-\rho} G_{El}(\rho)$

$$\longrightarrow \quad \frac{d^2 G_{El}}{d\rho^2} - 2\frac{dG_{El}}{d\rho} + \left[\frac{e^2 K}{E\rho} - \frac{l(l+1)}{\rho^2}\right] G_{El}(\rho) = 0$$

2nd order ODE for $y(x)$

$$\frac{d^2 y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\, y(x) = 0$$

· behavior near $x=0$

governing sol$^{n}$s

① $P(x)$ & $Q(x)$ are analytic @ $x=0$
Taylor expand dat boi
$P(x)$ & $Q(x)$ don't blow up

Th$^{m}$: unique sol$^{n}$ obeying $y(0) = a_0$ & $\frac{dy(0)}{dx} = a_1$ satisfying

of the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$, can find a series sol$^{n}$

OG: $\dfrac{d^2h}{dy^2} - \underbrace{2y}_{P(y)} \dfrac{dh}{dy} + \underbrace{(\mathcal{E}-1)}_{Q(y)} h(y) = 0$

both are analytic $y=0$ for P, constant for Q

$P(\rho) = -2$ , analytic @ $\rho = 0$

$Q(\rho) = \left(\dfrac{\rho^2 K}{E\rho} - \dfrac{\ell(\ell+1)}{\rho^2}\right)$ , not analytic @ $\rho = 0$ b/c of $\frac{1}{\rho}$ & $\frac{1}{\rho^2}$

New Theorem Check!

def$^n$: $x=0$ is a regular singular point

regular → nothing crazy

singular → diverging

of $\dfrac{d^2y}{dx^2} + P(y)\dfrac{dy}{dx} + Q(x)y = 0$

if $x P(x) = p_0 + p_1 \cdot x_1 + \dots$

$x^2 Q(x) = q_0 + q_1 \cdot x_1 + \dots$

if P(x) & Q(x) are analytic

P doesn't blow up worse than $\frac{1}{x}$

Q " $\frac{1}{x^2}$

so $\rho = 0$ is a regular singular point

th$^m$:

if $xP(x)$, $x^2 Q(x)$ are analytic there is at least one sol$^n$

of the form $y = x^m \sum_{n=0}^{\infty} a_n x^n$

$= x^m (a_0 + a_1 x + \dots)$

m can be fractional or negative

sol$^n$ → $m(m-1) + m p_0 + q_0 = 0$

2 roots: $m_1$ & $m_2$

if $m_1$ & $m_2$ $\in R$, and $m_2 \leq m_1$

sol$^n$: $y = x^{m_1} \sum_{n=0}^{\infty} a_n x^n$ RHB 16

Frobenius method

$\rho P(\rho) = -2\rho \qquad p_0 = 0$

$\rho^2 Q(\rho) = -l(l+1) + Q(\rho) \qquad q_0 = -l(l+1)$

$\rightarrow \quad m(m-1) - l(l+1) = 0$

2 roots: $\quad m = l+1, \ -l$

$$G_{El}(\rho) = \rho^{l+1} \sum_{k=0}^{\infty} C_k \rho^k$$

$$\frac{dG_{El}}{d\rho} = \sum_{k=0}^{\infty} (k+l+1) C_k \rho^{k+l}$$

$$\frac{d^2 G_{El}}{d\rho^2} = \sum_{k=0}^{\infty} (k+l+1)(k+l) C_k \rho^{k+l-1}$$

$$\sum_{k=0}^{\infty} \Big[ (k+l)(k+l+1) C_k \rho^{k+l-1} - 2(k+l+1) C_k \rho^{k+l} + \frac{\rho^2 K}{E} C_k \rho^{k+l} - l(l+1) C_k \rho^{k+l-1} \Big] = 0$$

as $\rho \to 0$, leading term is

$$\rho^{l-1} \big[ l(l+1) C_0 - l(l+1) C_0 \big] = 0$$

$k=0$

$$\sum_{k=0}^{\infty} \rho^{l-1} \Big[ (k+l)(k+l+1) C_k \rho^k - 2(k+l+1) C_k \rho^{k+1} + \frac{\rho^2 K}{E} C_k \rho^{k+1} - l(l+1) C_k \rho^k \Big] = 0$$

$\rho^{k+l-1} \qquad \rho^{l-1}$

$$\frac{\rho^{l-1}}{\rho^{k+l-1}} = \rho^{-k} \longrightarrow \infty \text{ as } \rho \to 0$$