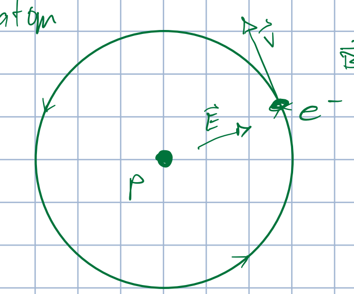
H atom



$\vec{B}$ in or out of board

e^- sees moving proton

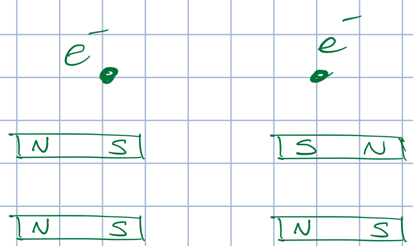
magnetic field $\quad \vec{B} \sim \vec{v} \times \vec{E} \quad \sim \vec{L}$

e^- has spin ½

magnetic dipole: $\vec{\mu} \propto \vec{S}$

$H_{interaction} \propto \vec{L} \cdot \vec{S} \cdot f(r)$ $\qquad [H_{int}, L_i] \neq 0 \qquad [H_{int}, S_i] \neq 0$

$[J_i, H_{int}] = 0 \qquad \vec{J} = \vec{S} + \vec{L}$



lower energy

higher energy

interaction depending on orientation of magnetic moment

$H_{int} = c\, \vec{S}_1 \cdot \vec{S}_2$

add ½

$|\tfrac{3}{2}, \tfrac{3}{2}\rangle = |1,1\rangle \otimes |+\rangle$

sub. ½

$|\tfrac{3}{2}, -\tfrac{3}{2}\rangle = |1,-1\rangle \otimes |-\rangle$

2 spin ½

$H_{spin} = \mathbb{C}^2 \otimes \mathbb{C}^2$

there are basis of states eigenf^k of $\vec{S}_1^{\,2}, \vec{S}_2^{\,2}, S_{1z}, S_{2z}$

$m = +½ \sim -½$

$S_z |s, m\rangle = m \frac{\hbar}{2} |s, m\rangle$

$|+\rangle_1 \otimes |+\rangle_2$

$|+\rangle_1 \otimes |-\rangle_2$

$|-\rangle_1 \otimes |+\rangle_2$

$|-\rangle_1 \otimes |-\rangle_2$

$S_z |\pm\rangle = \pm \frac{\hbar}{2} |\pm\rangle$

$S^2 |\pm\rangle = s(s+1)\hbar^2 |\pm\rangle = \frac{3}{4}\hbar^2 |\pm\rangle$ $\quad s = ½$

act on seperate vector spaces

problem: $\vec{S}_1 \cdot \vec{S}_2$ not diagonal in this basis

$(\vec{S}_1 \otimes \mathbb{1})(\mathbb{1} \otimes \vec{S}_2) = \sum_i (S_1)_i \otimes (S_2)_i = S_{1x} \otimes S_{2x} + S_{1y} \otimes S_{2y} + S_{1z} \otimes S_{2z}$

$[S_{1x}, S_{1z}] \neq 0$

want basis w/ $\vec{S}_1 \cdot \vec{S}_2$ diagonal

not diagonal

write $\vec{S} = \vec{S}_1 \otimes \mathbb{1} + \mathbb{1} \otimes \vec{S}_2 = \vec{S}_1 + \vec{S}_2$

$$\vec{S}\cdot\vec{S} = \vec{S}_1 \cdot \vec{S}_1 + \vec{S}_2 \cdot \vec{S}_2 + 2\vec{S}_1 \vec{S}_2 = \vec{S}^2 = \vec{S}_1^2 + \vec{S}_2^2 + 2\vec{S}_1\vec{S}_2$$

$$\vec{S}^2 - \vec{S}_1^2 - \vec{S}_2^2 = 2\vec{S}_1\vec{S}_2$$

$$\vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}\left(\vec{S}^2 - \vec{S}_1^2 - \vec{S}_2^2\right)$$

$\vec{S}_1\vec{S}_2$ will be basis if we can find basis that diagonalizes $\vec{S}^2$, $\vec{S}_1^2$, & $\vec{S}_2^2$

we know $\vec{S}^2$ & S_z work together

change of basis $(\vec{S}_1^2, \underline{S_{1z}}, \vec{S}_2^2, \underline{S_{2z}}) \to (\underline{\vec{S}^2}, \underline{S_z}, \vec{S}_1^2, \vec{S}_2^2)$

diagonal diagonal

b/c $\vec{S}_1$ & $\vec{S}_2$ obey this relation

"addition of angular momentum"

$[S_i, S_j] = \epsilon_{ijk} S_k i\hbar \implies$ theory of angular momentum applies to spin

$\vec{S}^2$ eigenvalues: $s(s+1)\hbar^2$ $\quad s = 0, \frac{1}{2}, 1, \dots$

S_z eigenvalues: $m\hbar$ $\quad m = -s, \dots, s$

spin 1/2 operators act on this space

$$\mathbb{C}^2 \otimes \mathbb{C}^2 = \left(V^{(1)}_{1/2} \otimes V^{(2)}_{1/2}\right) \overset{\text{name}}{=} V_{1/2 \otimes 1/2}$$

Basis of $V_{1/2\otimes 1/2}$

$|+\rangle_1 \otimes |+\rangle_2 = |++\rangle$ $\quad |-\rangle_1 \otimes |+\rangle_2 = |-+\rangle$

$|+\rangle_1 \otimes |-\rangle_2 = |+-\rangle$ $\quad |-\rangle_1 \otimes |-\rangle_2 = |--\rangle$

$$S_z = S_{1,z} \otimes \mathbb{1} + \mathbb{1} \otimes S_{2,z}$$

$$S_z |++\rangle = \frac{\hbar}{2}|++\rangle + \frac{\hbar}{2}|++\rangle = \hbar\, |++\rangle$$

$m=1$ state for S_z $\quad m = m_1 + m_2$

$|++\rangle$ must be a $s=1$ state?

$$S_z|+-\rangle = \left(\frac{\hbar}{2} - \frac{\hbar}{2}\right)|+-\rangle = 0$$

$|+\rangle =$

$$S_z|-+\rangle = \left(-\frac{\hbar}{2} + \frac{\hbar}{2}\right)|-+\rangle = 0$$

$$S_z|--\rangle = \left(-\frac{\hbar}{2} - \frac{\hbar}{2}\right)|--\rangle = -\hbar\,|--\rangle$$

in this basis

S_z eigenvalues are $-\hbar, 0, 0, \hbar$

$m = -s, \dots, s$ for fixed s: $s = 0 \to m = 0$ $\quad s = \frac{1}{2} \to m = -\frac{1}{2}, \frac{1}{2}$

$s = 1 \to m = -1, 0, 1$ $\quad s = \frac{3}{2} \to m = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$

add together to get m eigenvalues

Check $\vec{S}^2|++\rangle = s(s+1)\hbar^2|++\rangle$ w/ $s=1$

$$\vec{S}^2 = \vec{S}_1^2 + \vec{S}_2^2 + 2\vec{S}_1\cdot\vec{S}_2$$

$\to 2(S_{1z}S_{2z} + S_{1+}S_{2-} + S_{1-}S_{2+})$

tries to raise first plus

tries to raise second plus

acting on $|++\rangle$

raising a $|+\rangle$ gives 0, so these go away

$$= \vec{S}_1^2 + \vec{S}_2^2 + 2S_{1z}S_{2z}$$

$$S = S_1 + S_2 \qquad m = m_1 + m_2$$

$\vec{S}^2|++\rangle = 1(1+1)\hbar^2|++\rangle$

$S_z|++\rangle = \hbar|++\rangle$

$|s=1,m=1\rangle = |++\rangle$

$\vec{S}_1^2|++\rangle = \vec{S}_1^2\otimes\mathbb{1}|++\rangle = (\vec{S}_1^2|+\rangle)\otimes|+\rangle = s_1(s_1+1)\hbar^2|++\rangle$

$S_2^2|++\rangle = (\vec{S}_2^2\otimes\mathbb{1})(|+\rangle\otimes|+\rangle) = s_2(s_2+1)\hbar^2|++\rangle$

$S_{1z}S_{2z}|++\rangle = S_{1z}(m_1\hbar|++\rangle) = m_1m_2\hbar^2|++\rangle$

$\left(\frac12(\frac12+1)\hbar^2 + \frac12(\frac12+1)\hbar^2 + 2\cdot\frac12\cdot\frac12\hbar^2\right)|++\rangle$

$\left(\frac12\cdot\frac32 + \frac12\cdot\frac32 + \frac12\right)\hbar^2$

$\left(\frac34+\frac34+\frac24\right)\hbar^2 = \frac84\hbar^2 = 2\cdot\hbar^2$

$\vec{S}^2 = s(s+1)\hbar^2|++\rangle$

$s = s_1+s_2 \to \frac12+\frac12 = 1$

$\vec{S}^2|++\rangle = 1(1+1)\hbar^2|++\rangle$

$= 2\hbar^2|++\rangle$

$|s=1,m=0\rangle \propto S_-|s=1,m=1\rangle$

$S_-|s,m\rangle = \hbar\sqrt{s(s+1)-m(m-1)}\,|s,m-1\rangle$

$|s,m-1\rangle = \frac{1}{\hbar\sqrt{s(s+1)-m(m-1)}}\cdot S_-|s,m\rangle$

$|s=1,1-1\rangle = \frac{1}{\hbar\sqrt{1(1+1)-1(1-1)}}\left(S_-^{(1)}|1,1\rangle + S_-^{(2)}|1,1\rangle\right)$

$m=1$

$= \frac{1}{\hbar\sqrt{2-0}}\left(S_-^{(1)}|++\rangle + S_-^{(2)}|++\rangle\right)$

$= \frac{1}{\hbar\sqrt2}\cdot(\hbar|-+\rangle + \hbar|+-\rangle) = \frac{1}{\sqrt2}(|-+\rangle + |+-\rangle) = |s=1,m=0\rangle$

$|s=1,m-1\rangle = \frac{1}{\hbar\sqrt{s(1+s)-m(m-1)}}\,S_-|s=1,m\rangle = \frac{1}{\sqrt2\hbar}S_-|s=1,m=1\rangle = \frac{1}{\sqrt2\hbar}(S_-^{(1)}+S_-^{(2)})|++\rangle$

$= \frac{1}{\sqrt2}(|+-\rangle + |-+\rangle)$

$m=0$

$|s=1,m=-1\rangle$ apply S_- to $|s=1,m=0\rangle$

$|s=1,m=-1\rangle = \frac{1}{\hbar\sqrt{1(1+1)-0(0-1)}}\left(S_-^{(1)}+S_-^{(2)}\right)|s=1,m=0\rangle$

$= \frac{1}{\sqrt2\hbar}\left(S_-^{(1)}+S_-^{(2)}\right)\cdot\frac{1}{\sqrt2}(|+-\rangle + |-+\rangle)$

$= \frac{1}{\hbar 2}\left(\left(S_-^{(1)}+\cancel{S_-^{(2)}}\right)|+-\rangle + \left(\cancel{S_-^{(1)}}+S_-^{(2)}\right)|-+\rangle\right)$

$= \frac{1}{2\hbar}\left(\hbar|--\rangle + \hbar|--\rangle\right)$

$= \frac12\cdot 2|--\rangle = |--\rangle$

switch $-\to+$, $+\to-$

$|s=1,m=1\rangle = |++\rangle$

$|s=1,m=0\rangle = \frac{1}{\sqrt2}(|+-\rangle + |-+\rangle)$ V_1 symmetric

$|s=1,m=-1\rangle = |--\rangle$

orthogonal

$|s=0,m=0\rangle = \frac{1}{\sqrt2}(|+-\rangle - |-+\rangle)$ V_0 antisymmetric

eigenvalue zero

$S^2\frac{1}{\sqrt2}(|+-\rangle - |-+\rangle)$

$= \frac{1}{\sqrt2}(S^2|+-\rangle - S^2|-+\rangle)$

$= \frac{1}{\sqrt2}\hbar\left(\sqrt{0(0+1)-0(0-1)}\,|+-\rangle - \text{" " } |-+\rangle\right)$

$= \frac{1}{\sqrt2}\hbar\cdot 0 = 0$

$V_{\frac12\otimes\frac12} = V_1 \oplus V_0$

total, spin 1, spin 0

$U = V\oplus W$

$U = V + W$ no vectors in common

$\mathbb{R}^4 = \mathbb{R}^3\oplus\mathbb{R}^1$

$S_i = S_{1i} + S_{2i}$

triplet state mix under S_i, never goes to singlet

S_i act on $V_{1/2}^{(1)} \otimes V_{1/2}^{(2)}$ look like

$$S_i = \begin{pmatrix} \begin{pmatrix} 3\times 3 \end{pmatrix} & 0 \\ 0 & (1\times 1) \end{pmatrix}$$

3×3 → act on $s=1$ $= S_1 + S_2$ $|s=1, m\rangle$

(1×1) → act on $s=0$ $= S_1 + S_2$ $|s=0, m=0\rangle$

→ rotationally invariant

→ generators of angular momentum on total system

$H = \lambda \vec{S}_1 \cdot \vec{S}_2$ $\qquad [H, S_{1i}] \neq 0, \ [H, S_{2i}] \neq 0$

$[H, S_i] = 0$

find eigenvalues of H

doesn't compute w/ z components so triplet state isn't a basis

$$\lambda \vec{S}_1 \cdot \vec{S}_2 = \frac{\lambda}{2}\left(\vec{S}^2 - S_1^2 - S_2^2 \right)$$

$s_1 = 1/2 \qquad s_2 = 1/2$

$$= \frac{\lambda}{2}\left(s(s+1)\hbar^2 - \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2 - \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2 \right)$$

$|s=1, m\rangle$
$|s=0, m=0\rangle$ are eigenstates of total spin

$s=1$

$$\longrightarrow \frac{\lambda}{2}\left(2 - \frac{3}{4} - \frac{3}{4}\right)\hbar^2 = \frac{\lambda}{2}\hbar^2\left(\frac{1}{2}\right) = \frac{\lambda\hbar^2}{4}$$

3 eigenvalues $|s=1, m=-1,0,1\rangle$

multiplicity of 3

only depends on s, not s_1, s_2, m_1, m_2

$s=0$

$$\longrightarrow \frac{\lambda}{2}\left(0 - \frac{3}{2}\right)\hbar^2 = -\frac{3}{4}\lambda\hbar^2$$

$V_{1/2} \otimes V_{1/2} = V_1 \oplus V_0 \qquad S_i = \begin{pmatrix} \boxed{3\times 3} & \\ & \boxed{1\times 1} \end{pmatrix}$

$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$

tensor product of vector spaces for two spin 1/2 particles

decompose into vector space of total spin 1, total spin 0 in atomic physics

$\vec{J} = \vec{L} + \vec{S}$ acting on $\underbrace{L^2(\mathbb{R}^3)}_{\text{fixed } l} \otimes V_{1/2}$ $m = -l, -l+1, \ldots, l$

Basis of $\vec{L}^2 \; L_z \quad \vec{S}^2 \; S_z$ are diagonal

say we know eigenvalues of $\vec{L}^2$ & $\vec{S}^2$: l & s

$|l, m\rangle \otimes |+\rangle$
$|l, m\rangle \otimes |-\rangle$ $\qquad m = -l, \ldots, l$

$\underset{-l \to l}{(2l+1)} \times \underset{+ \text{ or } -}{(2)}$ space $= 4l+2$ space w/ fixed $\vec{L}^2 \; \vec{S}^2$ eigenvalues

look for eigenstates of $\vec{J}^2, J_z, \vec{L}^2 \; \vec{S}^2$ like going from $\vec{S}_1^2 S_{1z} \; \vec{S}_2^2 S_{2z} \to \vec{S}^2 S_z \; \vec{S}_1^2 \vec{S}_2^2$

start w/ state w/ largest m & s_z values

$$\underbrace{J_z \, |l, l\rangle \otimes |+\rangle = \underbrace{(l+\tfrac{1}{2})\hbar}\,|l, l\rangle \otimes |+\rangle}_{\text{eigenstate of } J_z \text{ w/ eigenvalue } (l+\frac{1}{2})\hbar}$$

$s = 1/2$

$= (L_z \otimes S_z)(|l,l\rangle \otimes |+\rangle)$
$= L_z |l, l\rangle \otimes S_z |s = 1/2, m = 1/2\rangle$
$= l\hbar |l, l\rangle \otimes \tfrac{1}{2}\hbar \, |+\rangle$
$= (l+1/2)\hbar \, (|l,l\rangle \otimes |+\rangle)$

this is the max J_z eigenvalue

$[J_i, J_j] = i\hbar J_k$

$|j, m_j\rangle$

$j = l + s \qquad m_j = m_l + m_s$

$\vec{J}^2 \, |j, m_j\rangle = j(j+1)\hbar^2 \, |j, m_j\rangle$

$J_z \, |j, m_j\rangle = m_j \hbar \, |j, m_j\rangle$

largest J_z value

$|j = l+1/2, \; m_j = l + 1/2\rangle = |l, l\rangle \otimes |+\rangle$

$|j = l+1/2, \; m_j = l - 1/2\rangle$

$\vdots$

$2j+1$

$\cancel{2l+1}$ states

can find each by acting w/ J_-

$2(l+\tfrac{1}{2})+1$

$2l+1+1$

$2l+2$

$|j = l+1/2, \; \underbrace{m_j = -(l+1/2)}_{\to \text{ min } J_z}\rangle = |l, -l\rangle \otimes |-\rangle$

Started w/ $(2l+1)\cdot 2 = 4l+2$ states. this accounts for $2l+2$

must be states w/ extra j values

$J_z = l-\frac{1}{2}$ $\quad |l, l-1\rangle \otimes |+\rangle$

$J_z = l-\frac{1}{2}$ $\quad |l, l\rangle \otimes |-\rangle$

2 states w/ $l-\frac{1}{2}$ eigenvalue

this is one linear combination here
must be an orthogonal one

new ladder should have max: $|j = l-\frac{1}{2}, m_j = l-\frac{1}{2}\rangle$

$\vdots$

$|j = l-\frac{1}{2}, m_j = -(l-\frac{1}{2})\rangle$

$\Big\}\ 2(l-\frac{1}{2})+1$ states: $2l$

$2(j_+)+1 + 2(j_-)+1$

$2(l+\frac{1}{2})+1+2(l-\frac{1}{2})+1$

$2l+1+1+2l\ \cancel{-1}+1$

$4l+2$

$V_{\frac{1}{2}} \otimes V_l = V_{l+\frac{1}{2}} \oplus V_{l-\frac{1}{2}}$

how to actually find states

① Start w/ max l & max spin: $|l=l, m=l\rangle \otimes |+\rangle = |j=l+\frac{1}{2}, m_j = l+\frac{1}{2}\rangle$

Apply J_- to get $|j = l+\frac{1}{2}, m_j = l-\frac{1}{2}\rangle$ ☆

again to get $|j = l+\frac{1}{2}, m_j = l-\frac{3}{2}\rangle$

$\vdots$

until get $|j = l+\frac{1}{2}, m_j = -l-\frac{1}{2}\rangle$

fill out the multiplet

② Find the state with $|j = l-\frac{1}{2}, m_j = l-\frac{1}{2}\rangle$ orthogonal to ☆

Apply J_- to fill out whole multiplet

$\vec{J}_1, \vec{J}_2$ are angular momentum operators could be any 2, 3, ...

$[J_i, J_j] = i\hbar \epsilon_{ijk} J_k$ is obeyed

let V_j be the $(2j+1)$ dimensional space spanned by all states $|j_1, j_1\rangle \dots |j_1, -j_1\rangle$

$V_j: |j_2, j_2\rangle \dots |j_2, -j_2\rangle$

$V_{j_1} \otimes V_{j_2}$ decompose to eigenstates of $\vec{J}^2$ & J_z

$= V_{(j_1+j_2)} \oplus V_{(j_1+j_2-1)} \oplus \dots \oplus V_{|j_1-j_2|}$

$V_{1/2} \otimes V_{1/2} = V_1 \oplus V_0$

$V_{3/2} \otimes V_{1/2} = V_2 \oplus V_1$

$V_2 \otimes V_2 = V_4 \oplus V_3 \oplus V_2 \oplus V_1 \oplus V_0$

$|j_1, m_1\rangle \otimes |j_2, m_2\rangle$

On $V_{j_1} \otimes V_{j_2}$ we have $|j_1, j_2, m_1, m_2\rangle$ basis

$$\mathbb{1} = \sum_{m_1} \sum_{m_2} |j_1, j_2, m_1, m_2\rangle \langle j_1, j_2, m_1, m_2|$$

$\Rightarrow \vec{J}_1^{\,2}, \vec{J}_2^{\,2}, J_{1z}, J_{2z}$

what about $|j, m, j_1, j_2\rangle$ to this basis?

$\vec{J}^2, \vec{J}_z, \vec{J}_1^{\,2}, \vec{J}_2^{\,2}$

$$|j, m, j_1, j_2\rangle = \sum_{m_1, m_2} |j_1, j_2, m_1, m_2\rangle \langle j_1, j_2, m_1, m_2 | j, m, j_1, j_2\rangle$$

$$= \sum_{m_1, m_2} \langle j_1, j_2, m_1, m_2 | j, m, j_1, j_2\rangle |j_1, j_2, m_1, m_2\rangle$$

express $\vec{J}^2$ & $\vec{J}_z$ eigenstates as linear combination of $\vec{J}_1, \vec{J}_2, J_{1z}, J_{2z}$ eigenstates

Clebsch - Gordon coefficients

$\vec{J}^2 |3/2, 3/2\rangle = \frac{3}{2}(\frac{3}{2}+1)\hbar^2 \, |3/2, 3/2\rangle$

$\vec{J}^2 J_- |3/2, 3/2\rangle = \frac{3}{2}(\frac{3}{2}+1)\hbar^2 \, J_- |3/2, 3/2\rangle$ $\quad [\vec{J}^2, J_-] = 0$

$|j, m\rangle$

$\uparrow \quad \uparrow$

$\vec{J}^2 \quad J_z$

$m = 3/2$ largest m

$\Rightarrow j = 3/2$

j gives m domain

$|3/2, 1/2\rangle = c_1 |1,1\rangle \otimes |-\rangle + c_2 |1,0\rangle \otimes |+\rangle$

state orthog to this which must be $|1/2, 1/2\rangle$

$$|1/2, 1/2\rangle \xrightarrow{J_-} |1/2, -1/2\rangle$$

$$V_1 \otimes V_{1/2} = V_{3/2} \oplus V_{1/2}$$

dim $3 \cdot 2 \qquad 4 + 2$

$$\begin{pmatrix} 6\times6 \end{pmatrix} = \begin{pmatrix} \boxed{4\times4} & \\ & \boxed{2\times2} \end{pmatrix}$$

$[J_i, J_j] = i\hbar\, \epsilon_{ijk} J_k \rightarrow \qquad J_\pm = J_1 \pm i J_2$

$[\vec{J}^2, J_z] = 0 \rightarrow$ can find simultaneous $\vec{J}^2, J_z$ eigenstates

$[\vec{J}^2, J_\pm] = 0 \rightarrow$ acting w/ $J_\pm$ does not change $\vec{J}^2$ eigenvalue

$|j, j\rangle$

$|j, j-1\rangle$

$\vdots$

$|j, -j\rangle$

$2j+1$ states

$J_+ |j, j\rangle = 0$

$J_- |j, -j\rangle = 0$

$\vec{J}^2 |j, m\rangle = \hbar^2 j(j+1) |j, m\rangle$

$J_z |j, m\rangle = \hbar\, m |j, m\rangle$

j integer: $|1, j\rangle \cdots |1, -j\rangle$

j half integer: no zero $|1/2, j\rangle \cdots |1/2, -j\rangle$

allowed j values: $0, 1/2, 1, 3/2, \ldots$

at fixed j, there are $2j+1$ states labeled by m. Form a subspace of total Hilbert space "w/ angular momentum j"

b/c $[H, \vec{J}^2] = 0 \rightarrow$ divide space of states to states w/ fixed $\vec{J}^2$ eigenvalue.

3 objects

① V_j — $2j+1$ dim. vector space over $\mathbb{C}$ w/ orthonormal basis

$|j, j\rangle, |j, j-1\rangle, \ldots, |j, -j+1\rangle, |j, -j\rangle$

$\langle j, m | j', m'\rangle = \delta_{m,m'}$

③ 3 $(2j+1) \times (2j+1)$ Hermitian matrix

$$(\vec{J}_i^{(j)})_{mm'} = \langle j, m | J_i | j, m' \rangle$$

obeying $[J_i^{(j)}, J_j^{(j)}] = i\hbar \epsilon_{ijk} J_k^{(j)}$

③ for each g element in group

$G = SO(3)$ $j = 0, 1, 2, \dots$
$G = SO(2)$ $j = 1/2, 3/2, \dots$

we have unitary matrix $\mathcal{D}_{mm'}^{(j)} = (e^{i\epsilon \hat{n} \hat{J}^{(j)}})_{mm'}$ for rotation by ϵ about $\hat{n}$

Hermitian $J_i^{(j)}$ & Unitary $\mathcal{D}^{(j)}$ act on V_j by...

$$J_z | j, m \rangle = \hbar m | j, m \rangle$$

$$J_\pm | j, m \rangle = \hbar \sqrt{j(j+1) - m(m\pm1)} \, | j, m \rangle$$

Can make new representations from old

direct sum & tensor product

↓

$$V = V_{j_1} \oplus V_{j_2} \oplus \dots \oplus V_{j_n}$$

decomposition of vector space into set of ortho. sub spaces s.t. any vector in V can be written as sum of V_j...

$$J_i = \begin{pmatrix} (J_i^{(j_1)}) & & & 0 \\ & (J_i^{(j_2)}) & & \\ & & \ddots & \\ 0 & & & (J_i^{(j_n)}) \end{pmatrix}$$

each $J_i^{j_k}$ is $(2j_k+1) \times (2j_k+1)$ matrix
can now be diagonalized.
can exponentiate like above

$$\mathcal{D} = \begin{pmatrix} \mathcal{D}^{(j_1)} & & & \\ & \mathcal{D}^{(j_2)} & & \\ & & \ddots & \\ & & & \mathcal{D}^{(j_n)} \end{pmatrix}$$

tensor product

$V_{j_1} \otimes V_{j_2}$

① is a vector space of dims $(2j_1+1) \times (2j_2+1)$

w/ basis vectors as tensor products of V_{j_1} & V_{j_2} basis vectors

$|j_1 m_1\rangle \otimes |j_2 m_2\rangle$

② provides $(2j_1+1) \times (2j_2+1)$ representation of Lie algebra of SO(3)

$$J_i^{(j_1 \otimes j_2)} = J_i^{(j_1)} \otimes \mathbb{1} + J_i^{(j_2)} \otimes \mathbb{1}$$

b/c $[J_i^{(j_1 \otimes j_2)}, J_j^{(j_1 \otimes j_2)}] = i\hbar \epsilon_{ijk} J_k^{(j_1 \otimes j_2)}$

③ provides group representation by $\mathcal{D}^{(j_1 \otimes j_2)} = \exp(-i\epsilon \hat{n} \cdot \vec{J}^{(j_1 \otimes j_2)})$

④ is reducible!

$$V_{j_1} \otimes V_{j_2} = V_{j_1+j_2} \oplus V_{j_1+j_2-1} \oplus \dots \oplus V_{|j_1-j_2|}$$

"addition of angular momentum" is working out decomposition by finding basis vectors of terms on RHS

$\mathbb{C}^2 \oplus \mathbb{C}^2$

$\begin{pmatrix} a \\ b \end{pmatrix} \quad \begin{pmatrix} c \\ d \end{pmatrix}$

$V = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$

$M_1 = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$ $\qquad m_1 = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}$

$N_1 = \begin{pmatrix} n_1 & 0 \\ 0 & n_2 \end{pmatrix}$

$$M_1 \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} m_1 \begin{pmatrix} a \\ b \end{pmatrix} \\ m_2 \begin{pmatrix} c \\ d \end{pmatrix} \end{pmatrix}$$

$$N_1 M_1 \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} n_1 m_1 \begin{pmatrix} a \\ b \end{pmatrix} \\ n_2 m_2 \begin{pmatrix} c \\ d \end{pmatrix} \end{pmatrix}$$ b/c diagonal

$V_1 =$ span of $\{ |11\rangle, |10\rangle, |1,-1\rangle \}$

$$(J_i^{(1)})_{mm'} = \langle 1m | J_i | 1m' \rangle$$

$$J_z |1m\rangle = \hbar m |1,m\rangle$$
$$J_- |1,1\rangle = \hbar \sqrt{2}\, |1,0\rangle$$
$$J_- |1,0\rangle = \hbar \sqrt{2}\, |1,-1\rangle$$
$$J_- |1,-1\rangle = 0$$

could compute $\mathcal{D}^{(1)} = e^{-i\epsilon \hat{n} \cdot \vec{J}^{(1)}}$

consider tensor product space

$V_1 \otimes V_1$ w/ basis $|1,m\rangle \otimes |1,m'\rangle$

total angular momentum is

$$J_i^{tot} = J_{i,1} \otimes \mathbb{1} + \mathbb{1} \otimes J_{i,2}$$

$$J_i^{tot} |1,m\rangle \otimes |1,m'\rangle = J_{i,1} |1,m\rangle \otimes |1,m'\rangle + |1,m\rangle \otimes J_{i,2} |1,m'\rangle$$

can define 9×9 matrices

$$\left(\langle 1,n | \otimes \langle 1,n' | \right) J_i^{tot} \left(|1,m\rangle \otimes |1,m'\rangle \right) = \left(J_i^{(1 \otimes 1)} \right)_{\substack{nn' \\ mm'}}$$

want to know eigenvalues of $(\vec{J}^{(1\otimes 1)})^2$ & $J_z^{(1\otimes 1)}$

$J_z^{(1\otimes 1)}$ eigenvalues are $(m+m')\hbar$ w/ $m, m' = -1, 0, 1$

m	m'	$m+m'$
1	1	2
1	0	1
1	-1	0
0	1	1
0	0	0
0	-1	-1
-1	1	0
-1	0	-1
-1	-1	-2

b/c we know that for a given j, $m = -j, \dots, j$ can group these

total ladder

$(2,1,0,-1,-2)$ $\quad j=2 \quad \equiv \quad j=2 \quad$ 5 rungs

$(1,0,-1)$ $\quad j=1 \quad \equiv \quad j=1 \quad$ 3

(0) $\quad j=0 \quad \equiv \quad j=0 \quad$ 1

$$V_1 \otimes V_1 = V_2 \oplus V_1 \oplus V_0$$

$m'+m = 2,-2$ must be in V_2

start w/ $|1,1\rangle \otimes |1,1\rangle$ $\quad m=m'=1$

act w/ J_-^{tot}

$$\left.\begin{aligned}
|2,2\rangle &= |1,1\rangle \otimes |1,1\rangle \\
|2,1\rangle &= \vdots \\
|2,0\rangle & \\
|2,-1\rangle & \\
|2,-1\rangle &= |1,-1\rangle \otimes |1,-1\rangle
\end{aligned}\right\} \text{basis of } V_2$$

remaining $m+m' = 1,-1$ states must be in V_1 & ortho to $m+m'=1,-1$ in V_2

same w/ $m+m'=0$

$$|2,2\rangle = |1,1\rangle \otimes |1,1\rangle$$

$$\underbrace{J_-^{tot}\,|2,2\rangle}_{\text{write explicitly}} = \cancel{\hbar}\sqrt{6-2}\;|2,1\rangle$$

$$= (J_{-,1}\,|1,1\rangle) \otimes |1,1\rangle + |1,1\rangle \otimes (J_{-,2}^{tot}\,|1,1\rangle)$$

$$= \cancel{\hbar}\sqrt{2}\,|1,0\rangle \otimes |1,1\rangle + |1,1\rangle \otimes \hbar\sqrt{2}\,|1,0\rangle$$

$$|2,1\rangle = \frac{1}{\sqrt{2}}\left(|1,0\rangle \otimes |1,1\rangle + |1,1\rangle \otimes |1,0\rangle\right)$$

do again

$$J_-^{tot}\,|2,1\rangle = \hbar\sqrt{6-0}\;|2,0\rangle$$

$$\rightarrow \quad |2,0\rangle = \frac{1}{\hbar\sqrt{6}}\left(J_{-1,1} \otimes \mathbb{1} + \mathbb{1} \otimes J_{-1,2}\right) \sqrt{2}\,\left(\qquad\right)$$

$$= \frac{1}{\sqrt{6}}\left(|1,-1\rangle \otimes |1,1\rangle + 2\,|1,0\rangle \otimes |1,0\rangle + |1,1\rangle \otimes |1,-1\rangle\right)$$

only gives 5 states. Now look @ $|2,1\rangle$ & make orthogonal for $|1,1\rangle$

$$|1,1\rangle = \frac{1}{\sqrt{2}}\left(|1,0\rangle \otimes |1,1\rangle - |1,1\rangle \otimes |1,0\rangle\right)$$

appy J_-^{tot}