$\Delta x \Delta p \geq \hbar/2$ SHO energy eigenstate $|n\rangle = 0, 1, 2, \dots$

$E_n = (n+1/2)\hbar\omega$

For $|n\rangle$ $\Delta x \Delta p = \frac{\hbar}{2}(2n+1)$

$n=0$ saturates the inequality

can we find states of SHO for which $\Delta x \Delta p = \hbar/2$

Generalize $a|0\rangle = 0$ $|0\rangle$ is an eigenvector of a w/ eigenvalue 0

claim: if we can find states $|\lambda\rangle$

$a|\lambda\rangle = \lambda|\lambda\rangle$ $\lambda \in \mathbb{C}$ then $\Delta x \Delta p = \hbar/2$ for these states

not Hermitian, so λ doesn't have to be $\mathbb{R}$

λ will have min. $\Delta x \Delta p$

$\langle\lambda| a^+ = \langle\lambda|\lambda^*$

assume $\langle\lambda|\lambda\rangle = 1$

looking @ expectation values

$\langle\lambda| a + a^+ |\lambda\rangle = \langle\lambda| \lambda + \lambda^* |\lambda\rangle = \lambda + \lambda^*$ $x \sim (a+a^+)$

$\langle\lambda| a - a^+ |\lambda\rangle = \langle\lambda| \lambda - \lambda^* |\lambda\rangle = \lambda - \lambda^*$ $p \sim (a - a^+)$

$\langle\lambda| (a+a^+)(a+a^+) |\lambda\rangle = \langle\lambda| (a+\lambda^*)(\lambda+a^*) |\lambda\rangle$

$= \langle\lambda| a\lambda + \lambda^*\lambda + aa^+ + \lambda^* a^+ |\lambda\rangle$

$= \langle\lambda| \lambda^2 + \lambda^*\lambda + \lambda^{*2} + a^+a + [a, a^+] |\lambda\rangle = (\lambda+\lambda^*)^2 + 1$

$\langle\lambda| (a-a^+)(a-a^+) |\lambda\rangle = (\lambda - \lambda^*)^2 - 1$

$$(\Delta x)^2 = \frac{\hbar}{2m\omega}\left(\langle\lambda|(a+a^+)^2|\lambda\rangle - (\langle\lambda|(a+a^+)|\lambda\rangle)^2\right)$$

$$= \frac{\hbar}{2m\omega}\left((\lambda+\lambda^*)^2 - 1 - (\lambda+\lambda^*)^2\right) = \frac{\hbar}{2m\omega}$$

$$(\Delta p)^2 = -\frac{m\omega\hbar}{2}\left((\lambda-\lambda^*)^2 - 1 - (\lambda-\lambda^*)^2\right) = \frac{m\omega\hbar}{2}$$

$$(\Delta x)^2(\Delta p)^2 = \frac{\hbar^2}{4} \longrightarrow \Delta x\, \Delta p = \frac{\hbar}{2}$$

$$|\lambda\rangle = c(\lambda)\ \exp(\lambda a^+)\ |0\rangle$$

claim $\quad a|\lambda\rangle = \lambda|\lambda\rangle$

$$a|\lambda\rangle = c(\lambda)\cdot a \sum_{n=0}^{\infty} \frac{\lambda^n (a^+)^n}{n!}|0\rangle \qquad |n\rangle = \frac{(a^+)^n}{\sqrt{n!}}|0\rangle$$

$$= c(\lambda)\cdot a \sum \frac{\lambda^n}{\sqrt{n!}}|n\rangle$$

$a|n\rangle = \sqrt{n}\,|n-1\rangle$

getrid of $n=0$ $\quad a|0\rangle = 0$

$$= c(\lambda)\cdot \sum_{n=1}^{\infty} \frac{\lambda^n \sqrt{n}}{\sqrt{n!}}|n-1\rangle$$

$$= c(\lambda) \sum \frac{\lambda^1\ \lambda^{n-1}}{\sqrt{(n-1)!}}|n-1\rangle$$

$$= \lambda e^{\lambda a^+}|0\rangle = \lambda|\lambda\rangle$$

physical interpretation of λ

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a+a^+)$$

$$\hat{p} = -i\sqrt{\frac{\hbar m\omega}{2}}(a-a^+)$$

$$\langle\lambda|\hat{x}|\lambda\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle\lambda|a+a^+|\lambda\rangle = \sqrt{\frac{\hbar}{2m\omega}}(\lambda+\lambda^*) = \sqrt{\frac{\hbar}{2m\omega}}\, 2\mathrm{Re}[\lambda]$$

$$\langle \lambda | \hat{p} | \lambda \rangle = \sqrt{\frac{\hbar m \omega}{2}} \; 2\,\mathrm{Im}[\lambda]$$

$$\lambda = \sqrt{\frac{m\omega}{2\hbar}} \langle \lambda | \hat{x} | \lambda \rangle + \frac{i}{\sqrt{2m\omega\hbar}} \langle \lambda | \hat{p} | \lambda \rangle$$

$|\lambda, t\rangle \rightarrow$ sol^n to time dependent SE

$$i\hbar \frac{\partial}{\partial t} |\lambda, t\rangle = H |\lambda, t\rangle$$

$$|\lambda\rangle = C(\lambda) \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}} |n\rangle$$

$E_n = (n + 1/2)\hbar\omega$

$H|n\rangle = E_n |n\rangle$

$$|\lambda, t\rangle = C(\lambda) \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}} e^{-iE_n t/\hbar} |n\rangle$$

$$= C(\lambda)\, e^{-i\omega t/2} \sum_{n=0}^{\infty} \frac{(\lambda e^{-i\omega t})^n}{\sqrt{n!}} |n\rangle$$

$$= C(\lambda)\, e^{-\frac{i\omega t}{2}} \exp(\lambda e^{-i\omega t} a^\dagger) |0\rangle$$

time dependence from $\lambda \rightarrow \lambda e^{-i\omega t}$

$\langle \hat{x} \rangle + i \langle \hat{p} \rangle$

Classically,

$x = x_0 \cos(\omega t)$

$p = m x_0 \omega \sin(\omega t)$

$$\langle \lambda | \lambda \rangle = 1$$

$$|\lambda\rangle = C(\lambda) \sum_{n=0}^{\infty} \frac{\lambda^n |n\rangle}{\sqrt{n!}}$$

$$\langle \lambda | = C^*(\lambda) \sum_{m=0}^{\infty} \frac{\langle m| \lambda^{*m}}{\sqrt{m!}}$$

$$\langle \lambda | \lambda \rangle = \sum_n \sum_m |C(\lambda)|^2 \langle m | \frac{\lambda^{*m}}{\sqrt{m!}} \frac{\lambda^n}{\sqrt{n!}} | n \rangle$$

orthonormal

$$= |C(\lambda)|^2 \sum_n \sum_m \frac{\lambda^{*n} \lambda^n}{n!}$$

$$= |C(\lambda)|^2 e^{\lambda^* \lambda} = |C(\lambda)|^2 e^{|\lambda|^2}$$

to be normalized, $C(\lambda) = e^{-|\lambda|^2/2}$

$$|\lambda\rangle = e^{-|\lambda|^2/2} e^{\lambda a^+} |0\rangle$$