$H = \hbar\omega(a^\dagger a + 1/2) = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$

$a = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right)$

$[a, a^\dagger] = 1$

$[H, a] = [\hbar\omega\, a^\dagger a + 1/2, a] = \hbar\omega[a^\dagger a, a] = \hbar\omega(\cancel{a^\dagger[a,a]}^{\,0} + [a^\dagger, a]a)$

$[AB, C] = ABC - CAB = ABC - ACB + ACB - CAB$

$= A[B C] + [A, C]B$

$= \hbar\omega\,(-1)\,a = -\hbar\omega\, a$

$[H, a^\dagger] = \hbar\omega a^\dagger$
$[H, a] = -\hbar\omega a$

can define $N = a^\dagger a$

$H = \hbar\omega(N + 1/2)$

$[N, a] = -a$
$[N, a^\dagger] = a^\dagger$
$[a, a^\dagger] = 1$

in QM, Hilbert space of states has positive norm

$|\psi\rangle \in \mathcal{H}$ obeys $\langle\psi|\psi\rangle \geq 0$, 0 iff $|\psi\rangle = 0$ 0 state vector

Start w/a normalized $|\psi\rangle$ ∍ $N|\psi\rangle = n|\psi\rangle$

$\rightarrow H|\psi\rangle = (n+1/2)\hbar\omega$

$H|n\rangle = \hbar\omega(n+1/2)|n\rangle$

consider $a|n\rangle$, $Na|n\rangle = (aN + Na - aN)|n\rangle$
$= (aN + [N,a])|n\rangle$
$= (aN + -a)|n\rangle$
$= aN|n\rangle - a|n\rangle$

$= a n|n\rangle - a|n\rangle$

$= (an - a)|n\rangle$

$= (n-1)a|n\rangle$

$N a^+|n\rangle = (a^+N + [N, a^+])|n\rangle$

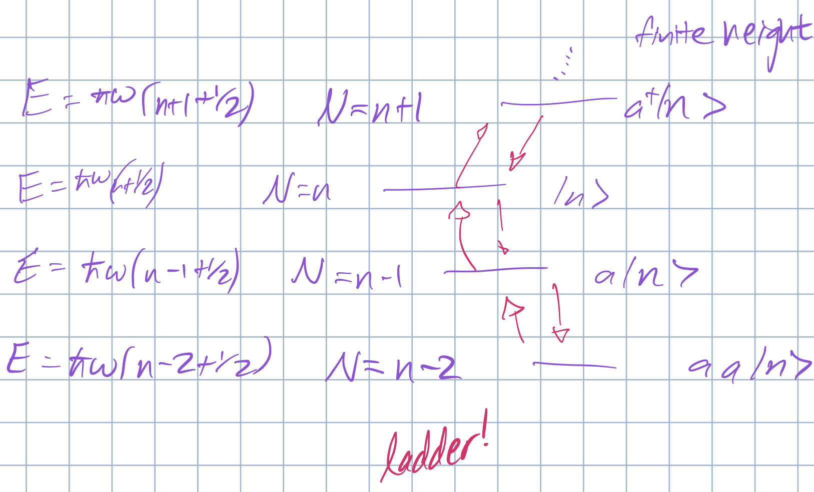
$= (a^+N + a^+)|n\rangle$

$= a^+N|n\rangle + a^+|n\rangle$

$= a^+n|n\rangle + a^+|n\rangle$

$= (a^+n + a^+)|n\rangle$

$= (n+1)a^+|n\rangle$



Check norm of states

$|a^+|n\rangle|^2 = \langle n|a a^+|n\rangle$

$= \langle n|[a, a^+] + a^+a|n\rangle$

$= \langle n|\; 1 + N\; |n\rangle$

$= \langle n|(|n\rangle + N|n\rangle)$

$= \langle n|(|n\rangle + n|n\rangle)$

$= \langle n|\; 1+n\; |n\rangle$

$= (1+n)\langle n|n\rangle = n+1$ $\leftarrow$ positive if $n+1>0$

$|a|n\rangle|^2 = \langle n|a^+a|n\rangle = n$ $\leftarrow$ positive if $n>0$

$a|n\rangle = |n-1\rangle$

$|aa|n\rangle|^2 = |a|n-1\rangle|^2 = n-1$

ladder ends @ ground state $|0\rangle$

$a|0\rangle = 0\cdot|0\rangle = 0$

only happens if n is integer

$E_n = (n+\frac{1}{2})\hbar\omega \qquad n = 0, 1, 2, 3, \ldots$

Start w/ $|0\rangle$ $\langle 0|0\rangle = 1$

norm of $a^+|0\rangle$ is $\langle 0|aa^+|0\rangle = \langle 0|a^+a + [a,a^+]|0\rangle$

$= \langle 0|1|0\rangle = 1$

$|1\rangle = a^+|0\rangle$

how about $a^+a^+|0\rangle$?

$|a|n\rangle|^2 = n-1$

$\langle 0|aaa^+a^+|0\rangle = \langle 1|aa^+|1\rangle$

$= \langle 1|a^+a + [a,a^+]|1\rangle$

$= \langle 1|N+1|1\rangle = 2$

$|0\rangle \to$ not 0 vector, is the SHO ground state

$|1\rangle = a^+|0\rangle$

$|2\rangle = \frac{1}{\sqrt{2}} a^+|1\rangle = \frac{1}{\sqrt{2}} a^+a^+|0\rangle$

$\vdots$

$|n\rangle = \frac{1}{\sqrt{n!}} (a^+)^n |0\rangle = \frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{(n-1)!}} (a^+)^n |0\rangle = \frac{1}{\sqrt{n}} |n-1\rangle$

$|2\rangle = \frac{1}{\sqrt{2}} \cdot \frac{1}{}$

$a|n\rangle = \langle n|a^+a|n\rangle$

$\sqrt{n} |n-1\rangle$

$\langle n-1|n-1\rangle n$

Start w/ ground state & look @ x component

$\langle x|0\rangle = \phi_0(x)$

SHO ground state found by SE?

$a|0\rangle = 0$

$\langle x|a|0\rangle = 0 = \langle x|\sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + i\frac{\hat{p}}{m\omega})|0\rangle$

$x \to x$ $\hat{p} = -i\hbar\frac{d}{dx}$

$0 = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{\hbar}{2m}\frac{d}{dx}\right)\underbrace{\langle x|0\rangle}_{\phi_0(x)}$

$\left(x + \frac{\hbar}{2m}\frac{d}{dx}\right)\phi_0(x) = 0$ first ODE

$$\frac{d\psi_0}{dx} = \frac{-m\omega x}{\hbar}\psi_0(x)$$

$$\psi_0(x) = C e^{-\frac{m\omega x^2}{2\hbar}}$$

$$\int |\psi_0(x)|^2 dx = 1 \rightarrow C = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

$$|1\rangle = a^\dagger|0\rangle$$

$$\phi_1(x) = \langle x|1\rangle = \langle x|a^\dagger|0\rangle$$

$$= \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{\hbar}{m\omega}\frac{d}{dx}\right)\psi_0(x)$$

by writing $a^\dagger$ in terms of $\hat{x}$ & $\hat{p}$

$$= \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{\hbar}{m\omega}\frac{d}{dx}\right)\cdot\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\cdot e^{-\frac{m\omega x^2}{2\hbar}}$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}\; x\, e^{-\frac{m\omega x^2}{2\hbar}}$$

$$\phi_n(x) = \langle x|\frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$$

$$= \frac{1}{\sqrt{n!}}\left(\frac{m\omega}{2\hbar}\right)^{1/2}\left(x - \frac{\hbar}{m\omega}\frac{d}{dx}\right)^n \phi_0(x)$$