SHO

$$\phi_E^{asym} = A e^{\pm y/2}$$

toss out $+y/2$ b/c it diverges

$$\phi_E(y) = h(y)\, e^{-y/2}$$ **Ansatz** - trial form / guess

demand growth not changing asymptotic behavior

$$\frac{d\phi_E}{dy} = \left(\frac{dh}{dy} - yh\right) e^{-y/2}$$

$$\frac{d^2\phi_E}{dy^2} = \left(\frac{d^2h}{dy^2} - 2y\frac{dh}{dy} + (y^2-1)h\right) e^{-y/2}$$

$$\frac{d^2\phi_E}{dy^2} - y^2\phi_E = -\varepsilon\phi_E$$

$$\xrightarrow{\text{math}} \quad \frac{d^2h}{dy^2} - 2y\frac{dh}{dy} + (\varepsilon - 1)h(y) = 0$$

Series solutions!

$$h(y) = \sum_{n=0}^{\infty} a_n y^n$$ Taylor expand around $y=0$

$$\frac{dh}{dy} = \sum_{n=0}^{\infty} n a_n y^{n-1}$$ n is a dummy variable

$$= a_1 + 2a_2 y^1 + \dots$$

$$= \sum_{n=0}^{\infty} (n+1) a_{n+1} y^n$$ same thing

$$\frac{d^2h}{dy^2} = \sum_{n=0}^{\infty} n(n-1)\, a_n\, y^{n-2} =$$

$$= 2a_2 + 6a_3 y + \dots$$

$$= \sum_{n=0}^{\infty} (n+1)(n+2)\, a_{n+2}\, y^n$$

$$\sum_{n=0}^{\infty} (n+1)(n+2)\, a_{n+2}\, y^n \quad - \quad 2\, y \cdot \sum_{n=0}^{\infty} n\, a_n\, y^{n-1} + (\varepsilon - 1) \sum_{n=0}^{\infty} a_n y^n = 0$$

$$\sum_{n=0}^{\infty} (n+1)(n+2)\, a_{n+2}\, y^n \quad - \quad 2\, y\, n\, a_n\, y^{n-1} + (\varepsilon - 1)\, a_n\, y^n = 0$$

$$\sum_{n=0}^{\infty} (n+1)(n+2)\, a_{n+2}\, y^n \quad - \quad 2\, n\, a_n\, y^n + (\varepsilon - 1)\, a_n\, y^n = 0$$

$$\sum_{n=0}^{\infty} y^n \left[(n+1)(n+2)\, a_{n+2} - 2\, n\, a_n + (\varepsilon - 1)\, a_n \right] = 0$$

true for all y $\quad (n+1)(n+2)\, a_{n+2} - 2\, n\, a_n + (\varepsilon - 1)\, a_n = 0$

$$\longrightarrow \quad a_{n+2} = a_n \left(\frac{2n+1-\varepsilon}{(n+1)(n+2)} \right)$$

a_0 fixes $a_2, a_4, \ldots$

a_1 fixes $a_3, a_5, \ldots$

2 kinds of sol$^{\underline{n}}$'s for a_{n+2} depending on ε

① $\varepsilon \neq 2n+1 \qquad n \in$ integers

infinite # of nonzero a_n behaves as $e^{x/2}$ as $y \to \infty$

bad ☹

② $\varepsilon == 2n+1 \qquad$ TRUE

suppose $\varepsilon = 5 \qquad$ or $n=2 \qquad \ni a_1 = 0 \quad \ni$ arbitrary a_0

$a_2 = \frac{1-\varepsilon}{2} a_0 = -2a_0$

$a_4 = \frac{5-\varepsilon}{12} a_2 = 0$

$a_6 = (\quad) a_4 = 0$

$a_8 = \quad = 0$

$\vdots$

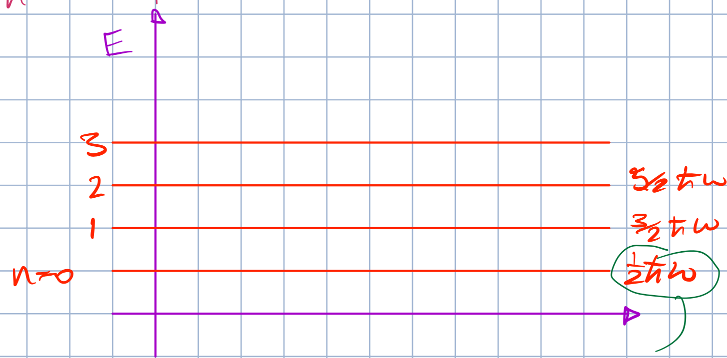
the series terminates

2 classes of sol'ns w/ $h(y)$ being finite order polynomial

① $a_1 = 0,\ a_0 \neq 0,\quad \mathcal{E} = 1, 5, 9, 13, \dots \qquad n = 0, 2, \dots$

② $a_1 \neq 0,\ a_0 = 0,\quad \mathcal{E} = 3, 7, 11, 15, \dots \qquad n = 1, 3, \dots$

$$\mathcal{E} = 2n+1 \longrightarrow E_n = \left(n + \tfrac{1}{2}\right)\hbar\omega$$



lowest energy state

classically, lowest is 0

<u>n=0</u>

$h = a_0 \quad , \quad \phi_0(y) = a_0 e^{-y^2/2}$

normalize & write in original variables

$$\phi_0(y) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\int_{-\infty}^{\infty} |\phi_0(x)|^2 dx = 1$$

ground state

<u>n=1</u> first excited state

$h = a_1 y \quad , \quad \phi_1(y) = a_1 y\, e^{-y^2/2}$

$$\phi_1(y) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{\frac{2m\omega}{\hbar}}\, x\, e^{-\frac{m\omega}{2\hbar}x^2}$$

$h(y)$ is polynomial of degree n for $\mathcal{E} = 2n+1$

only even powers of y for n even

" " odd " " odd

$E_n = (n + \tfrac{1}{2})\hbar\omega$

$\phi_E \to \phi_n \qquad \phi_n(-x) = (-1)^n \phi_n(x)$

parity operator

$[H, P] = 0 \quad \longrightarrow$ simultaneous H, P eigenf^{n}'s

$$\phi_n(x) = \underset{\substack{\uparrow \\ \text{constant f}^{\underline{n}} \text{ of n Griffiths 2.85}}}{C_n} \underbrace{H_n(y)}_{\text{Hermite polynomials}} e^{-y^2/2} \qquad y = \sqrt{\frac{m\omega}{\hbar}}\, x$$

"her meat"

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 = \hbar\omega\left(a^{+}a + \frac{1}{2}\right)$$

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)$$

$$a^{+} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)$$

$$a^{+}a = \frac{m\omega}{2\hbar}\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right)\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)$$

cross terms $\quad \frac{m\omega}{2\hbar}\left(\frac{i}{m\omega}\left(\underbrace{\hat{x}\hat{p} - \hat{p}\hat{x}}_{-i\hbar}\right)\right)$

Algebraic Structure involving $[a^{+}, a]$, $[H, a]$, $[H, a^{+}]$

$H|\psi\rangle = E|\psi\rangle$

$a|\psi\rangle = E - \hbar\omega$ annihilator operator

$a^{+}|\psi\rangle = E + \hbar\omega$ creation operator

$a^{+}\|\psi\rangle$	$E + \hbar\omega$
$H\|\psi\rangle$	E
$a\|\psi\rangle$	$E - \hbar\omega$

$$[a, a^+] = \frac{m\omega}{2\hbar}\left[\hat{x} + i\frac{\hat{p}}{m\omega}, \hat{x} - i\frac{\hat{p}}{m\omega}\right]$$

$$= \frac{m\omega}{2\hbar}\left(-\frac{i}{m\omega}[\hat{x},\hat{p}] + \frac{i}{m\omega}[\hat{p},\hat{x}]\right)$$

$$= \frac{m\omega}{2\hbar}\left(-\frac{2i}{m\omega}[\hat{x},p]\right)$$

$$= \frac{\cancel{m}\cancel{\omega}}{\cancel{2}\cancel{\hbar}}\left(-\frac{\cancel{2}i}{\cancel{m}\cancel{\omega}}\cdot i\cancel{\hbar}\right)$$

$$= -i^2 = 1$$

$$[a^+, a] = -1$$

$$\phi_0(y) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\phi_0(y) = h(y)\, e^{-y^2/2}$$

$$\frac{-y^2}{2} = -\frac{1}{2}\left(\sqrt{\frac{m\omega}{\hbar}}\,x\right)^2 = -\frac{m\omega}{2\hbar}x^2$$

$$e^{-y^2/2} = e^{-\frac{m\omega}{2\hbar}x^2}$$

for $n=0$, we have a_0, so we choose $a_1=0$ & $\varepsilon = 2n+1 = 1 \to a_{n+2} = \frac{2n+1-1}{(n+1)(n+2)}a_0 \overset{n=0}{=} 0$

$\to a_2 = 0$ terminates series

$$h(y) = \sum a_n y^n$$

$$= \sum_{\text{even}} a_n y(n) + \sum_{\text{odd}} a_n y(n) \to 0$$

$$= a_0 + \sum_{\text{even } n>0} a_n y^n \to 0$$

$$= a_0$$

$$\phi(x) = a_0\, e^{-\frac{m\omega}{2\hbar}x^2}$$

Gaussian integrals to find $a_0 = 1$