QM Math

orthonormal basis (ON) for V, $|i\rangle$, $\langle i|j\rangle = \delta_{ij}$

Identity operator $\mathbb{1}|v\rangle = |v\rangle$ $\qquad \mathbb{1} = \sum_i |i\rangle\langle i|$

$|w\rangle\langle v|$ $\longrightarrow$ $V \to V$ or $V^* \to V^*$ $\quad$ V^*: dual vector space, space of linear f'ts

$\langle v|$: linear maps $V \to \mathbb{F}$ $\qquad$ vector to scalar

O ? : Operators from $V \to V$ $\qquad$ vector to vector

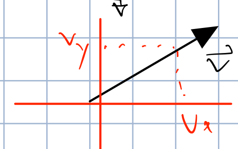
A linear operator $O : V \to V$ obeys

① $O(a|v\rangle) = a\,O|v\rangle$ $\qquad$ scales proportionally

② $O(a|v\rangle + b|w\rangle) = a\,O|v\rangle + b\,O|w\rangle$ $\qquad$ respects addition of vectors

$|v\rangle$, O $\qquad$ abstract

like



$\vec{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$

Vector is Indpt. of coords.

Suppose $|v\rangle \in V$, an ON basis $|i\rangle$

$|v\rangle = \mathbb{1}|v\rangle = \sum_i |i\rangle\langle i|v\rangle$ $\qquad v_i = \langle i|v\rangle$

$= \sum_i |i\rangle\, v_i$

$\to$ components organize like a column vector

specific to basis $\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$

$O : V \to V$ $\qquad O = \mathbb{1}\, O\, \mathbb{1} = \sum_i |i\rangle\langle i|\, O \sum_j |j\rangle\langle j|$

$= \sum_{ij} |i\rangle \underbrace{\langle i|O|j\rangle}_{\text{operate on } j} \langle j|$

compute inner product $\mathcal{O}$ element of matrix

$|w\rangle = \mathcal{O}\,|v\rangle$
basis indpt.

$\langle k|w\rangle = w_k = \langle k|\mathcal{O}|v\rangle$

$= \langle k| \sum_{ij} |i\rangle\langle i|\mathcal{O}|j\rangle\langle j|v\rangle$

we know $\langle k|i\rangle = \delta_{ik}$

$= \sum_j \langle k|\mathcal{O}|j\rangle\langle j|v\rangle$

$= \mathcal{O}_{kj}\, v_j$

$$\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} \mathcal{O}_{11} & \mathcal{O}_{12} & \cdots & \mathcal{O}_{1N} \\ \vdots & & & \vdots \\ \mathcal{O}_{m1} & \cdots & \cdots & \mathcal{O}_{mN} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix}$$

$\langle w|$ → dual vector, linear map or fk $V \to \mathbb{F}(\mathbb{F})$

$\langle w|v\rangle$ action of w on $|v\rangle$

$|v\rangle\langle w|$ Vector to a vector

Suppose we have 2 vector spaces V, W

2 ways of making new vector spaces from V & W

① direct sum $V \oplus W$ dimension $(V \oplus W) = \dim(V) + \dim(W)$

② tensor product $V \otimes W$ dimension $(V \otimes W) = \dim(V) \cdot \dim(W)$

photons we used $\mathbb{C}^2$

$2+2 = 2 \cdot 2$

lol both are 4-D

Example $V = \mathbb{C}^2$ $W = \mathbb{C}^3$ defined over same scalars ($\mathbb{C}$, $\mathbb{R}$ ...)

$V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ $W = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$

how to make new vector space?

Stack! $\begin{pmatrix} v_1 \\ v_2 \\ w_1 \\ w_2 \\ w_3 \end{pmatrix}$

does this obey axioms?

multiply by scalars $\alpha \begin{pmatrix} v_1 \\ v_2 \\ w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} \alpha v_1 \\ \alpha v_2 \\ \alpha w_1 \\ \alpha w_2 \\ \alpha w_3 \end{pmatrix}$

prime

addition $\begin{pmatrix} v_1 \\ v_2 \\ w_1 \\ w_2 \\ w_3 \end{pmatrix} + \begin{pmatrix} v_1' \\ v_2' \\ w_1' \\ w_2' \\ w_3' \end{pmatrix} = \begin{pmatrix} v_1 + v_1' \\ v_2 + v_2' \\ w_1 + w_1' \\ w_2 + w_2' \\ w_3 + w_3' \end{pmatrix}$

Let U be a vector space & V, W vector subspaces of U

if $U = V + W$ & $V \cap W = \{0\}$

V intersect w/ W is zero vector

then we say U is direct sum of V & W $\longrightarrow U = V \oplus W$

$U = \mathbb{R}^3$ $V = \text{span}\{\hat{e}_x\}$ $W = \text{span}\{\hat{e}_y, \hat{e}_z\}$

$\mathbb{R}^3 = \mathbb{R} + \mathbb{R}^2$ $\mathbb{R}$ $\mathbb{R}^2$

Tensor Product

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \qquad w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

want to multiply components of vectors, not vectors themselves

6 combos: $v_1 w_1$, $v_1 w_2$, $v_1 w_3$,
$v_2 w_1$, $v_2 w_2$, $v_2 w_3$

$(m \times n) \cdot (n \times p) = (m \times p)$
matrix

v is 2 by 1 $\qquad$ w is 3 by 1 $\qquad$ cant do 2 by 1 • 3 by 1
can do 2 by 1 · 1 by 3! so let's take w's transpose

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} (w_1, \; w_2, \; w_3) = \begin{pmatrix} v_1 w_1 & v_1 w_2 & v_1 w_3 \\ v_2 w_1 & v_2 w_2 & v_2 w_3 \end{pmatrix}$$

$$\| \qquad v \otimes w$$

NOT SAME AS CROSS PRODUCT

define $V \otimes W$ to be space of all linear combinations of $v \otimes w$ for $v \in V$, $w \in W$

$w_1 = 1$ & rest $= 0$, $w_2 = 0$ $\qquad$ $w_2 = 0$

$V \otimes W$ contains $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ....

every matrix of 1 & rest zero

$V \otimes W$ contains $\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & \alpha_6 \end{pmatrix}$ for all $\alpha_i \in \mathbb{C}$

all 2×3 matrices

vectors M & N

M, 2×3 complex matrix
N, " "
$M + N$ " "
αM " " , $\alpha \in \mathbb{C}$

inner product! $\qquad \langle M | N \rangle = Tr(M^{+} N)$

choose basis $|i\rangle$ $i = 1, 2$ for V
" $|a\rangle$ $a = 1, 2, 3$ for W

any element of $V \otimes W$ can be written as $\sum_i \sum_a \alpha_{ia} \underbrace{|i\rangle \otimes |a\rangle}_{\text{basis for } V \otimes W} = |\alpha\rangle$

linear combination

$|\beta\rangle = \sum_j \sum_b \beta_{jb} |j\rangle \otimes |b\rangle$

compute $\langle \beta | \alpha \rangle = \text{Trace}(\beta^+ \alpha)$

$\sum_{jb} \langle j| \bullet \langle b| \; \beta^*_{jb} = \langle \beta|$

on V $\langle j|i\rangle = \delta_{ij}$

$\langle \beta | \alpha \rangle = \sum_{jb} \langle j| \otimes \langle b| \beta^*_{jb} \quad \sum_{ia} \alpha_{ia} |i\rangle \otimes |a\rangle$

on W

$\langle b|a\rangle$ produces scalar $= \delta_{ab}$

$= \sum_{jb} \sum_{ia} \beta^*_{jb} \alpha_{ia} \delta_{ij} \delta_{ab}$ $a = b$

$= \sum_{jb} \beta^*_{jb} \alpha_{ib} \delta_{ij}$ $i = j$

$= \sum_{jb} \beta^*_{jb} \alpha_{jb}$

$|i\rangle_V \otimes |a\rangle_W$

describe original space w/ subscript

$= \sum_{jb} (\beta^+)_{bj} \alpha_{jb}$ ① multiply ② sum diagonal

$= Tr(\beta^+ \alpha) = (Tr(\alpha^+ \beta))^*$

need $\mathbb{R}$ norm

$z \in \mathbb{C}$

$z = x + iy$

$z \cdot z =$ norm ? no

$z \cdot z^* = (x+iy)(x-iy)$