group (G) – set of objects $\{g_1, g_2, \ldots\}$

finite or infinite

& a composition law

$\circ : G \times G \to G$

obeys axioms

① associativity $\quad g_i \circ (g_j \circ g_k) = (g_i \circ g_j) \circ g_k$

② identity $\quad$ there exists $e \in G$, $e \circ g = g \circ e = g$, all $g \in G$

③ inverse $\quad$ if $g \in G$, there exists $g^{-1} \in G$ s.t. $g^{-1} \circ g = g \circ g^{-1} = e$

examples

① $G = \{e\}$ $\qquad e \circ e = e$

② $G = \{1, -1\}$ $\qquad \circ = *$

$|\circ| = 1 \cdot 1 = 1$

$1 \cdot -1 = -1$

$-1 \cdot 1 = -1$

$-1 \cdot -1 = 1$

$(-1)^{-1} = -1$

$(1)^{-1} = 1$

③ $G = \{0, 1\}$ $\qquad \circ = + \bmod 2$

$0 + 0 = 0$

$1 + 0 = 1$

$0 + 1 = 1$

$1 + 1 = 2$

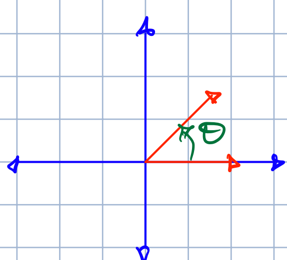
② $\to$ ③

$1 \to 0$

$-1 \to 1$

$* \to + \bmod 2$

④ Rotations of plane



$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

G = set of all $R(\theta)$

$0 \le \theta \le 2\pi$

$\circ$ = matrix multiplication

$SO(2, \mathbb{R})$

2x2 matrices $\quad \mathbb{R}$ elements

orthogonal $R^T = R^{-1}$
special : $\det(R) = 1$

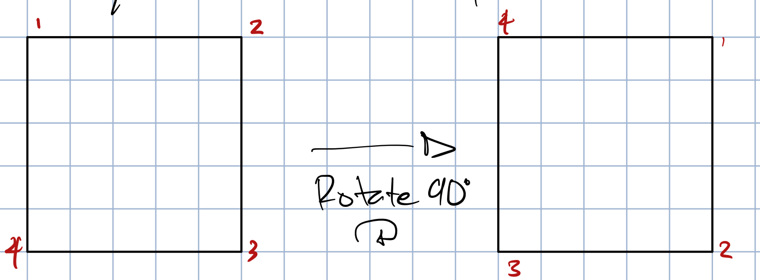
finite $G = \{1, -1\}$ $\mathbb{Z}_2$
infinite $SO(2, \mathbb{R})$ $R(\theta)$

if $g_i \circ g_j = g_j \circ g_i$ for all g_i, g_j $\rightarrow$ group is Abelian

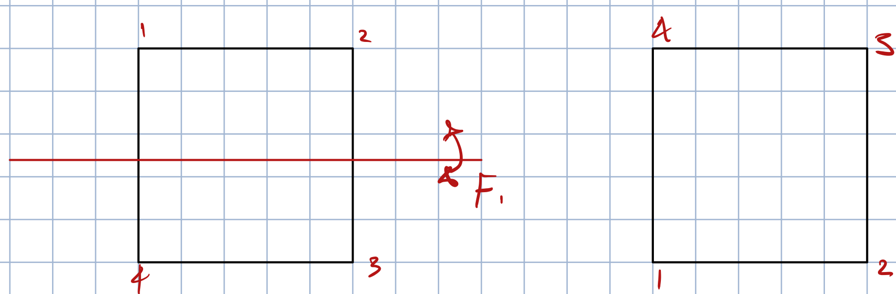
else $\rightarrow$ non Abelian

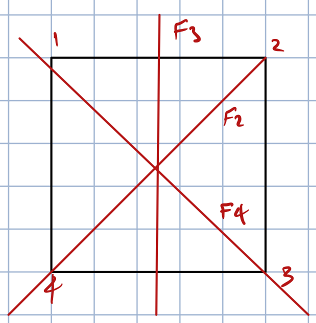
a non abelian - finite group:

group of symmetries of a square



identity e, R, R^2, R^3, $R^4 = e$

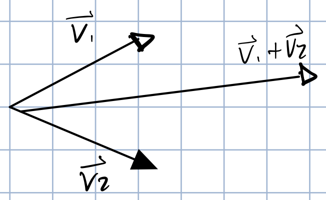


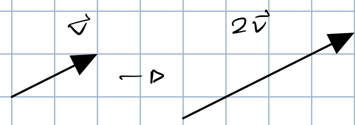


G has 4 rotations & 4 flips

b/c $FR \neq RF \rightarrow$ nonabelian

Vector Space





set of $V = \{v_1, v_2, \dots\}$

1. Addition rule $\quad v_i, v_j \in V \quad , \quad v_i + v_j \in V$
2. $(V, +)$ is an Abelian group $\quad$ objects are vectors & composition is addition
3. for scalar $\alpha, \beta, \dots, \in \mathbb{F}$ $\quad (\mathbb{R}, \mathbb{C}$ usually$)$

$\mathbb{F}$: field

$$\alpha(v_i + v_j) = \alpha v_i + \alpha v_j$$

$$(\alpha + \beta) v_i = \alpha v_i + \beta v_i$$

$$\alpha(\beta v_i) = \alpha \beta v_i$$

$$\underset{\text{vector}}{\vec{F}} = \underset{\text{scalar}}{m}\ \underset{\text{vector}}{\vec{a}}$$

scalars

in QM: $\quad |\psi\rangle = \alpha |v_1\rangle + \beta |v_2\rangle \qquad \alpha, \beta \in \mathbb{C}$

vectors

Dirac

given vector space V, there is always define a dual vector space of linear functions $f: V \to \mathbb{F}$

- $f(v_1+v_2) = f(v_1) + f(v_2)$
- $f(\alpha v) = \alpha f(v)$

space of all linear $f^{\underline{s}}$'s on V is dual vector space

- $(f_1+f_2)(v) = f_1(v) + f_2(v)$
- $(\alpha f)(v) = \alpha f(v)$

in Physics, almost always have vector spaces w/ an inner product $\underbrace{(\ , \ )}_{\text{2 vectors} \to \text{scalar}}$

axioms of inner products

$(v_1, v_2+v_3) = (v_1, v_2) + (v_1, v_3)$

$(v_1, a v_2) = a(v_1, v_2)$

$(v_1, v_2) = (v_2, v_1)^*$

$(v_1, v_1) \geq 0$ $\qquad$ $=0$ only if v is 0 vector

examples

$V = \mathbb{R}^3$

$(\vec{v}, \vec{w}) = \vec{v} \cdot \vec{w}$

if $v, w \in \mathbb{C}^n$

$(v, w) = \sum_i v_i^* w_i$

$v \in V$ & an inner product

$v \longrightarrow (v, \cdot) \xrightarrow{\text{Dirac}} \langle v|$

put any $w \in V$ here

$(v, \cdot)(w) = (v, w)$ $\qquad$ defines a linear $f^{\underline{n}}$: $V \to \mathbb{F}$

$(v, w) \xrightarrow{\text{Dirac}} \langle v | w \rangle$

$|v\rangle \longrightarrow \langle v|$ acts on w by $\langle v|w\rangle$

vector dualvector

in $\mathbb{C}^n$ + choose a basis

$$|v\rangle = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

$$\langle v| = (v_1^* \ \ v_2^* , \dots)$$

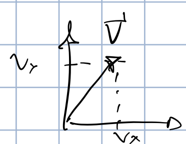
$$\langle v|w\rangle = (v_1^*, v_2^*, \dots)\begin{pmatrix} w_1 \\ \vdots \\ \ \end{pmatrix} = \sum_{i=1}^{n} v_i^* w_i$$

if $|v\rangle \in V$ $\quad \langle v|v\rangle = 1$

$P_v = |v\rangle\langle v|$ Hermitian projection operator onto $|v\rangle$

$$P_v^+ = (|v\rangle)^+(|v\rangle)^+ = |v\rangle\langle v|$$

$$P_v^2 = |v\rangle\langle v|v\rangle\langle v| = |v\rangle\langle v|$$



$$\vec{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

$|\psi\rangle$ which just are

often choose basis $|i\rangle$ $\quad \langle i|j\rangle = \delta_{ij}$ components $\psi_i = \langle i|v\rangle$