Midterm
24 hours
11:30 am Friday

Analogy $\mathcal{H} = \mathbb{C}^N$

$\mathcal{H} = L^2[0,L]$ space of f^n's
$0 = f(0) = f(L)$, $\int_0^L |f(x)|^2 dx$ is finite
→ time indpt. SE in particle in box

$|v\rangle$, $|w\rangle$ $\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$
$\alpha|v\rangle + |w\rangle$

if $f(x)$ & $g(x) \in L^2[0,L]$ so is $\alpha f(x) + \beta g(x)$, $\alpha, \beta \in \mathbb{C}$

inner product:
$\langle v | w \rangle = v^{+} \cdot w$

$\langle f | g \rangle = \int_0^L f^*(x)\, g(x)\, dx$

orthonormal basis:
$|i\rangle$, $i = 1, \dots, N$ $\quad \langle i | j \rangle = \delta_{ij}$

$|v\rangle = \sum_i v_i |i\rangle$
$v_i = \langle i | v \rangle$

$\phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$

$\langle \phi_n | \phi_m \rangle = \delta_{nm}$ $\quad n = 1, \dots, \infty$

$f(x) \in L^2[0,L] = \sum_n c_n \phi_n(x)$

Fourier Series $\quad c_n = \langle \phi_n | f \rangle$

vector ⇝ f^n

physical observables ~

$O = O^{+}$ operates on $\mathbb{C}^N$

$O_{ij} = \langle i | O | j \rangle = \langle j | O | i \rangle^*$

hermitian: $O_{ij} = O_{ji}^*$
wrt. ON basis

also have impt. Hermitian operators
→ $\hat{x}$, $\hat{p}$

$\hat{H} = \frac{\hat{p}^2}{2m} + V(x)$

is $\hat{p} = -i\hbar \frac{d}{dx}$ Hermitian?

is $\langle \phi_n | \hat{p} | \phi_m \rangle = \langle \phi_m | \hat{p} | \phi_n \rangle^*$

$\langle \phi_n | \hat{p} | \phi_m \rangle = \int_0^L \phi_n^*(x) \cdot -i\hbar \frac{d}{dx} \phi_m(x)\, dx$

goal to have $\frac{d}{dx}$ on first

$= -i\hbar \int_0^L \phi_n^*(x) \cdot \frac{d}{dx}\left(\phi_m(x)\right) dx$

product rule backwards

$$= -i\hbar \int_0^L \left[\frac{d}{dx}\left(\phi_n^*(x)\, \phi_m(x)\right) - \frac{d}{dx}(\phi_n^*(x)) \cdot \phi_m(x) \right] dx$$

$\downarrow \int \frac{d}{dx}$

$$= -i\hbar \underbrace{\left[\phi_n^*(x)\, \phi_m(x) \Big|_0^L \right]} + i\hbar \left[\int_0^L \frac{d}{dx}\left(\phi_n^*(x)\right) \phi_m(x) \right] dx$$

vanishes at L & 0
$\phi_n(0) = 0 = \phi_n(L) = \phi_m(0) = \phi_m(L)$

$$= \int_0^L \phi_m(x) \cdot i\hbar \frac{d}{dx} \cdot \phi_n^*(x) = \langle \phi_m | \hat{p} | \phi_n \rangle^*$$

$\hat{x}, \hat{p}, \hat{H} \sim \infty \times \infty$ matrices

$$X_{nm} = \langle \phi_n | x | \phi_m \rangle = \int_0^L \phi_n^*(x) \cdot x \cdot \phi_m(x)\, dx$$

$$p_{nm} = \langle \phi_n | p | \phi_m \rangle = \int_0^L \phi_n^*(x) \cdot \left(-i\hbar \frac{d}{dx}\right) \phi_m(x)\, dx$$

$$H_{nm} = \langle \phi_n | H | \phi_m \rangle = \langle \phi_n | E_m | \phi_m \rangle =$$
$$= E_m \delta_{nm} = \begin{pmatrix} E_1 & & 0 \\ & E_2 & \\ 0 & & \ddots \end{pmatrix}$$

H always diagonal

$$[\hat{x}, \hat{p}] \equiv \hat{x}\hat{p} - \hat{p}\hat{x}$$

good to have this act on a f^{n} of x

consider $(\hat{x}\hat{p} - \hat{p}\hat{x}) f(x)$

$$\hat{x}\hat{p} f(x) = \hat{x}\left(-i\hbar \frac{df}{dx}\right) = -i\hbar x \frac{df}{dx}$$

$$\hat{p}\hat{x} f(x) = -i\hbar \frac{d}{dx}\left(x f(x)\right) = -i\hbar f(x) - i\hbar x \frac{df}{dx}$$

cancel out

$$(\hat{x}\hat{p} - \hat{p}\hat{x}) f(x) = i\hbar f(x)$$

$$[\hat{x}, \hat{p}] = i\hbar$$

$$\Delta p \Delta x = \hbar/2$$

General sol$^{\underline{n}}$

$$\psi(x,t) = \sum_{n=1}^{\infty} c_n \phi_n(x) e^{\frac{-iE_nt}{\hbar}}$$

QM ~ CM

CM eq$^{\underline{n}}$ of motion hold in avg sense in QM

Function

Poisson bracket

$f(p,x)$
$g(p,x)$

$$\{f,g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial x}$$

$$\{f,g\} = -\{g,f\}$$ antisymmetric

$$\{f, \{g,h\}\} + \{g, \{h,f\}\} + \{h, \{f,g\}\} = 0$$ Jacobi identity

$$\{x,x\} = \{p,p\} = 0$$

$$\{x,p\} = 1$$

$$\frac{dx}{dt} = \{x, H\}$$

$$\frac{dp}{dt} = \{p, H\}$$

$p = mv$

$\frac{dp}{dt} = -\frac{dv}{dx}$

Hermitian $\quad O|\lambda\rangle = \lambda|\lambda\rangle$

$\langle\psi|O|\psi\rangle$ = expectation value of O (in state $|\psi\rangle$)

$$= \langle\psi|O\, \underbrace{\sum_i |\lambda\rangle\langle\lambda|}_{\mathbb{1}}\, \psi\rangle = \sum_i \lambda_i \langle\psi|\lambda_i\rangle\langle\lambda_i|\psi\rangle$$

$$= \sum_i \lambda_i |\langle\lambda_i|\psi\rangle|^2$$

$$= \underbrace{\sum_i \lambda_i p_i}_{\text{avg over probabilities}} = \langle\lambda_i\rangle = \langle O\rangle = \langle\psi|O|\psi\rangle$$

probabilities

measuring diff. eigenvalues

sum quantity · probability