Dirac delta f^n

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \text{``}\infty\text{''} & x = 0 \end{cases}$$

$$\int_{-\infty}^{\infty} f(x')\, \delta(x - x')\, dx' = f(x)$$

$f(x) \in L^2(\mathbb{R})$

can be any numbers, is a dummy variable to integrate over

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx$$

not integrated over

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \tilde{f}(k)\, dk$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx'} f(x')\, dx' \right) dk$$

exchange order of integration

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx'\, f(x') \left[\int_{-\infty}^{\infty} e^{ik(x-x')}\, dk' \right]$$

$$= \int_{-\infty}^{\infty} dx'\, f(x') \underbrace{\left[\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')}\, dk' \right]}_{= \delta(x - x')}$$

Suppose we have

$$\int_{-\infty}^{\infty} f(x)\ \delta(ax)\,dx \qquad x' = ax \qquad \frac{dx'}{dx} = a \qquad dx' = a\,dx$$

$$= \int_{-\infty}^{\infty} f\!\left(\tfrac{x'}{a}\right) \delta(x')\ \frac{dx'}{a}$$

where diracdelta vanishes

$$= \frac{f(0)}{a} \qquad \delta(ax) = \frac{\delta(x)}{|a|}$$

Fourier Transform is change of basis in Hilbert space $L^2(\mathbb{R})$

basis in Hilbert space

$|x\rangle$ - position eigenstate $\qquad \hat{x}|x\rangle = x\,|x\rangle$

$|p\rangle$ - momentum eigenstate $\qquad \hat{p}|p\rangle = p\,|p\rangle$

F.T. is on one or other

there should be analog of ① $\langle \lambda_i | \lambda_j \rangle = \delta_{ij}$ ② $\mathbb{1} = \sum_i |\lambda_i\rangle\langle\lambda_i|$

$$\langle x | x' \rangle = \delta(x - x')$$
$$\langle p | p' \rangle = \delta(p - p')$$
$$\mathbb{1} = \int dx\ |x\rangle\langle x| = \int dp\ |p\rangle\langle p|$$

Write an element of $L^2(\mathbb{R})$ as $|f\rangle$ $\sim$ $|v\rangle$ is vector in V

$\langle i|v\rangle$ - components in ON basis $|i\rangle$

components of f

$$\langle x | f \rangle = f(x)$$
$$\langle p | f \rangle = \tilde{f}(p)$$

$\langle x | p \rangle$ - momentum eigenstate in coordinate basis. $\sim$ $p(x)$

$$\langle x | p \rangle = e^{\frac{ipx}{\hbar}} \cdot \frac{1}{\sqrt{2\pi\hbar}} \qquad \hat{p} = -i\hbar\frac{d}{dx}$$

convenient normalization factor

$$f(x) = \langle x|f\rangle = \langle x|\mathbb{1}|f\rangle = \langle x|\overbrace{\int_{-\infty}^{\infty} dp\, |p\rangle\langle p|}^{\mathbb{1}}\, f\rangle$$

$$= \int_{-\infty}^{\infty} dp\, \langle x|p\rangle \underbrace{\langle p|f\rangle}_{\tilde{f}(p)}$$

$$= \int_{-\infty}^{\infty} dp \quad e^{\frac{ipx}{\hbar}} \cdot \frac{1}{\sqrt{2\pi\hbar}}\, \tilde{f}(p)$$

okay, cool. what about $\tilde{f}(p)$?

$$\tilde{f}(p) = \langle p|f\rangle = \langle p|\mathbb{1}|f\rangle = \langle p|\overbrace{\int_{-\infty}^{\infty} dx\, |x\rangle\langle x|}^{\mathbb{1}}\, f\rangle$$

$$= \int_{-\infty}^{\infty} dx \quad \langle p|x\rangle \langle x|f\rangle$$

$$= \int_{-\infty}^{\infty} dx \quad \langle x|p\rangle^{*} \langle x|f\rangle$$

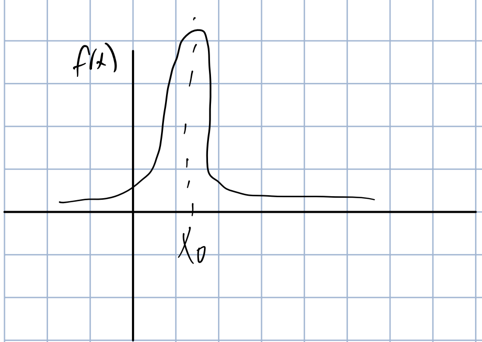
$$= \int_{-\infty}^{\infty} dx \quad e^{\frac{-ipx}{\hbar}} \cdot \frac{1}{\sqrt{2\pi\hbar}}\, f(x)$$

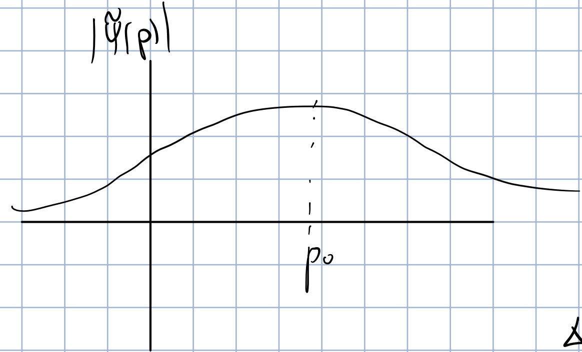
$p = \hbar k$

$$\tilde{f}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dx \quad e^{\frac{-ipx}{\hbar}}\, f(x) \qquad e^{-ikx}$$

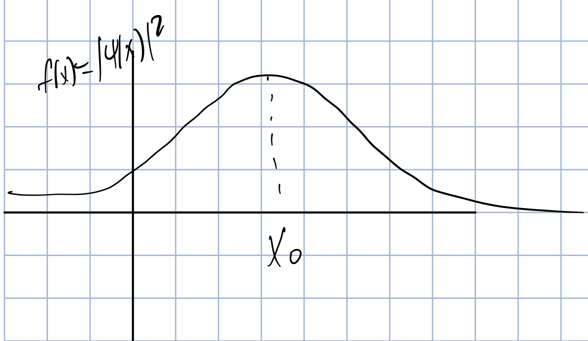
$$f(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dp \quad e^{\frac{ipx}{\hbar}}\, \tilde{f}(p) \qquad e^{ikx}$$

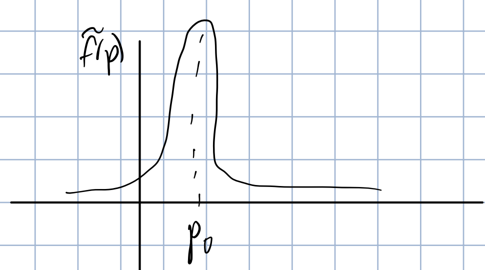
$$\int \frac{dp}{2\pi\hbar}\, e^{\frac{ip(x-x')}{\hbar}} = \delta(x-x') = \int \frac{dk}{2\pi}\, e^{ik(x-x')}$$





$$\Delta x \Delta p \geq \frac{\hbar}{2}$$





express $\langle\psi| e^{i\hat{p}a/\hbar} |\psi\rangle$ in terms of $\psi(x)$ & $\tilde{\psi}(p)$

$$\langle\psi| e^{\frac{i\hat{p}a}{\hbar}} \mathbb{1} |\psi\rangle$$

$$\langle\psi| e^{\frac{i\hat{p}a}{\hbar}} \int_{-\infty}^{\infty} dp\, |p\rangle\langle p|\psi\rangle$$

$$\int_{-\infty}^{\infty} dp\, e^{\frac{i\hat{p}a}{\hbar}} \langle\psi|p\rangle \langle p|\psi\rangle$$

$$\int_{-\infty}^{\infty} dp\; e^{\frac{i\hat{p}a}{\hbar}} \langle p|\psi\rangle^* \langle p|\psi\rangle$$

$$\int_{-\infty}^{\infty} dp\; e^{\frac{i\hat{p}a}{\hbar}} \underline{\tilde{\psi}^*(p)}\; \underline{\tilde{\psi}(p)}$$

from before

$$\underline{\tilde{\psi}(p)} = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \psi(x)\,dx$$

$$\underline{\tilde{\psi}^*(p)} = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{\frac{ipx'}{\hbar}} \psi^*(x')\,dx'$$

$$\frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx'\; e^{\frac{ipa}{\hbar} - \frac{ipx}{\hbar} + \frac{ipx'}{\hbar}} \psi(x)\, \psi^*(x')$$

quick aside

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dp\; e^{\frac{ip}{\hbar}(a-x+x')} = \hbar\, \delta(a-x+x')$$

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx'\; \psi(x)\, \psi^*(x')\, \underbrace{\delta(a-x+x')}_{\to\, 0 \;@\; x' = x-a}$$

$$\int_{-\infty}^{\infty} dx\; \psi(x)\, \psi^*(x-a)$$