uncertainty principle

prob. distribution, some quantity A

$$(\Delta A)^2 = \langle (A - \langle A \rangle)^2 \rangle = \langle A^2 - \langle A \rangle^2 \rangle$$

Apply to QM   state $|\psi\rangle$, operator $\hat{O}$

$\langle \psi | \hat{O} | \psi \rangle$ avg of $\hat{O}$

$\langle \psi | \hat{O}^2 | \psi \rangle$ avg of $\hat{O}^2$

$$(\Delta O)^2 = \langle \psi | \hat{O}^2 | \psi \rangle - (\langle \psi | \hat{O}^2 | \psi \rangle)^2$$

if $|\psi\rangle = |\lambda_i\rangle$ ✓ $\hat{O}|\lambda_i\rangle = \lambda_i |\lambda_i\rangle$

$$\hat{O}|\lambda_i\rangle = \lambda_i |\lambda_i\rangle$$

$$\hat{O}^2 |\lambda_i\rangle = \hat{O}\cdot\hat{O}|\lambda_i\rangle = \lambda_i \hat{O} |\lambda_i\rangle = \lambda_i^2 |\lambda_i\rangle$$

$$(\Delta O)^2 = \langle \lambda_i | \lambda_i^2 | \lambda_i \rangle - (\langle \lambda_i | \lambda_i | \lambda_i \rangle)^2 = \lambda_i^2 - \lambda_i^2 = 0$$

uncertainty in eigenstate is zero

$\Delta p \Delta x \geq \hbar/2$ any quantum state

$\hat{a} = \hat{p} - \langle \psi | \hat{p} | \psi \rangle \mathbb{1}$, $\hat{a}^+ = \hat{a}$

$\hat{\beta} = \hat{x} - \langle \psi | \hat{x} | \psi \rangle \mathbb{1}$ $\hat{\beta}^+ = \hat{\beta}$

$$|\phi\rangle = \hat{a}|\psi\rangle + i\lambda \hat{\beta}|\psi\rangle$$

$\langle \phi | \phi \rangle \geq 0$ positive definite norm in Hilbert space

$$\langle \phi | \phi \rangle = \left(\langle \psi | \hat{a}^+ - i \overset{\mathbb{R}}{\lambda} \langle \psi | \hat{\beta}^+ \right)\left(\hat{a}|\psi\rangle + i\lambda \hat{\beta}|\psi\rangle \right)$$

$$= \left(\langle \psi | \hat{a} - i\lambda \langle \psi | \hat{\beta} \right)\left(\hat{a}|\psi\rangle + i\lambda \hat{\beta}|\psi\rangle \right)$$

$$= \langle \psi | \hat{a}\hat{a} | \psi \rangle + \lambda^2 \langle \psi | \hat{\beta}\hat{\beta} | \psi \rangle + i\lambda \langle \psi | \hat{a}\hat{\beta} | \psi \rangle - i\lambda \langle \psi | \hat{\beta}\hat{a} | \psi \rangle$$

$$= \langle\psi|\,\hat{a}^2|\psi\rangle + \lambda^2 \langle\psi|\hat{\beta}^2|\psi\rangle + i\lambda\,\langle\psi|\,\hat{a}\hat{\beta} - \hat{\beta}\hat{a}|\psi\rangle$$

$$\hat{a}\hat{\beta} - \hat{\beta}\hat{a} = \hat{p}\hat{x} - \hat{x}\hat{p} = -i\hbar$$

$$= \langle\psi|\,\hat{a}^2|\psi\rangle + \lambda^2 \langle\psi|\hat{\beta}^2|\psi\rangle + \lambda\hbar\,\langle\psi|\psi\rangle$$

$$= \langle\psi|\,\hat{a}^2|\psi\rangle + \lambda^2 \langle\psi|\hat{\beta}^2|\psi\rangle + \lambda\hbar \quad \geq 0$$

let's choose λ to minimize : $\lambda_{min} = -\frac{\hbar}{2\langle\psi|\hat{\beta}^2|\psi\rangle}$ $\qquad \frac{d}{d\lambda}$

$$= \langle\psi|\,\hat{a}^2|\psi\rangle + \frac{\hbar^2}{4(\langle\psi|\hat{\beta}^2|\psi\rangle)^2}\langle\psi|\hat{\beta}^2|\psi\rangle - \frac{\hbar^2}{2\langle\psi|\hat{\beta}^2|\psi\rangle}$$

$$= \langle\psi|\hat{a}^2|\psi\rangle + \frac{\hbar^2}{4\langle\psi|\hat{\beta}^2|\psi\rangle} - \frac{\hbar^2}{2\langle\psi|\hat{\beta}^2|\psi\rangle} \geq 0$$

$$\langle\psi|\hat{a}^2|\psi\rangle \geq \frac{\hbar^2}{2\langle\psi|\hat{\beta}^2|\psi\rangle} - \frac{\hbar^2}{4\langle\psi|\hat{\beta}^2|\psi\rangle}$$

$$\langle\psi|\hat{a}^2|\psi\rangle \geq \frac{\hbar^2}{4\langle\psi|\hat{\beta}^2|\psi\rangle}$$

$$\langle\psi|\,\hat{a}^2|\psi\rangle\,\langle\psi|\hat{\beta}^2|\psi\rangle \geq \frac{\hbar^2}{4}$$

$$(\Delta p)^2 \quad (\Delta x)^2 \geq \hbar^2/4$$

$$\Delta p\ \Delta x \geq \hbar/2$$

$$(\Delta\hat{A})^2(\Delta\hat{B})^2 \geq \left(\frac{1}{2i}\ \langle\psi|\,[\hat{A},\hat{B}]|\psi\rangle\right)^2$$

Free particle in 1-D

$H = \frac{\hat{p}^2}{2m}$ no $V(x)$

$\psi(x,t)$ demand $\int_{-\infty}^{\infty} P(x,t)dx = \int_{-\infty}^{\infty} |\psi(x,t)|^2 dx = 1$

an eigenstate of $\hat{p}$ will be eigenstate of $\hat{H}$

$\underbrace{-i\hbar \frac{d}{dx}}_{\hat{p}} f_p(x) = \underbrace{p}_{\text{eigenvalue}} f(x) \longrightarrow f_p(x) = e^{ipx/\hbar}$

$\hat{H} f_p(x) = \frac{\hat{p}^2}{2m} f_p(x)$

$\psi(x,t) = e^{ipx/\hbar} \cdot e^{-i\left(\frac{p^2}{2m}\right)t/\hbar}$ $\longrightarrow$ not normalizable (not square integrable)

$|\psi(x,t)|^2 = 1$ so $\int_{-\infty}^{\infty} \psi(x,t)dx = \int_{-\infty}^{\infty} 1 = \infty$

$\longrightarrow$ not in $L^2(\mathbb{R})$ not in a Hilbert space of square integrable f^n's on $[-\infty,\infty]$

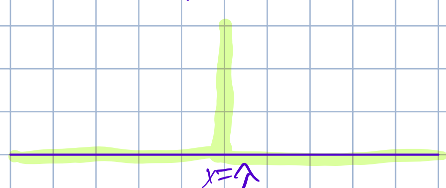
unphysical!

what about $\hat{x}$ eigenstates?

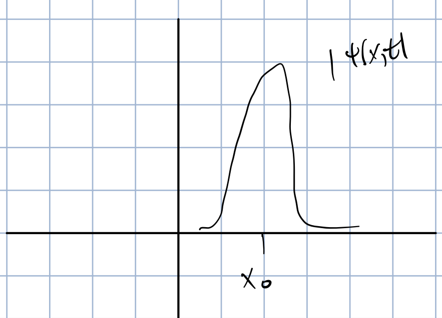
$\hat{x} f_\lambda(x) = \lambda f_\lambda(x)$

$\quad ||$

$x f_\lambda(x) = \lambda f_\lambda(x) \Longrightarrow f_\lambda(x) = 0$ except @ $x = \lambda$



$\longrightarrow \int_{-\infty}^{\infty} |f(x)|^2 = 0$



Fourier transform as it appears in QM as change of basis in a vector space

$f(x) \in L^2(\mathbb{R})$

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(k)\, e^{ikx}\, dk$$