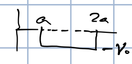
Ehrenfest's thm

derive uncertainty principle

1-d potential   free particle $U(x)$, SHO $V(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 x^2$, finite square well 

Ehrenfest's Thm

$\frac{dx}{dt} = \{x, H\}$ $\frac{dp}{dt} = \{p, H\}$ Poisson brackets

$\langle \psi | \underset{\text{Hermitian}}{\hat{O}} | \psi \rangle$ = expectation value = $\sum \lambda_i p_i$

$|\psi(x,t)\rangle$ obeying $\hat{H}\psi(x,t) = i\hbar \frac{\partial \psi}{\partial t}$

$\langle x | \psi(t) \rangle = \psi(x,t)$

$\langle \psi(t) | \hat{x} | \psi(t) \rangle = \int \psi^*(x,t) \cdot x \cdot \psi(x,t)\, dx$

limits depend on problem $\int_0^L$ particle in box $\int_{-\infty}^{\infty}$ other finite potentials

assume it vanishes @ limits

want to compute its time derivative

$\frac{d}{dt}\langle \psi(t) | \hat{x} | \psi(t) \rangle = \int \left(\frac{\partial \psi^*}{\partial t} \hat{x} \psi + \psi^* \hat{x} \frac{\partial \psi}{\partial t} \right) dx$

$\frac{\partial \psi}{\partial t} = \frac{-i}{\hbar} \hat{H}\psi$

$\frac{\partial \psi^*}{\partial t} = \frac{i}{\hbar}(\hat{H}\psi)^*$

$= \int \left(\frac{i}{\hbar} (\hat{H}\psi)^* \hat{x} \psi - \frac{i}{\hbar} \psi^* \hat{x} \hat{H} \psi \right) dx$

$\hat{H} = \hat{H}^*$ b/c

& b/c integration by parts

$= \int \frac{i}{\hbar} \psi^* \hat{H} \hat{x} \psi - \frac{i}{\hbar} \psi^* \hat{x} \hat{H} \psi \, dx$

$= \int \psi^* \left(\frac{i}{\hbar} \hat{H}\hat{x} - \frac{i}{\hbar} \hat{x}\hat{H} \right) \psi \, dx$

$= \int \psi^* \left(-\frac{i}{\hbar} (\hat{x}\hat{H} - \hat{H}\hat{x}) \right) \psi \, dx$

$= \int \psi^*(x,t) \left(\frac{-i}{\hbar} [\hat{x}, \hat{H}] \right) \psi(x) \, dx$

$= \langle \psi(t) | \frac{-i}{\hbar} [\hat{x}, \hat{H}] \psi(t) \rangle$

$\frac{d\langle x \rangle}{dt} = \frac{-i}{\hbar} \langle [\hat{x}, \hat{H}] \rangle$

$\{f, g\} \longrightarrow -\frac{i}{\hbar}[\hat{f}, \hat{g}]$ & avg. $\langle \psi |$ $| \psi \rangle$

$\frac{d\langle p \rangle}{dt} = -\frac{i}{\hbar} \langle [\hat{p}, \hat{H}] \rangle$

$\xrightarrow{\text{math}} \frac{d}{dt}\langle \psi(t) | \hat{p} | \psi(t) \rangle = \frac{-i}{\hbar} \langle \psi(t) | [\hat{p}, \hat{H}] | \psi(t) \rangle$

suppose $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$

Check this math

$$[\hat{x},\hat{H}] = \hat{x}\hat{H} - \hat{H}\hat{x} = \frac{1}{2m}[\hat{x},\hat{p}^2] = \frac{1}{2m}(\hat{x}\hat{p}\hat{p} - \hat{p}\hat{p}\hat{x})$$

$[\hat{x},\hat{p}] = i\hbar$ $\quad \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar$ $\quad$ or $\quad \hat{x}\hat{p} = \hat{p}\hat{x} + i\hbar$

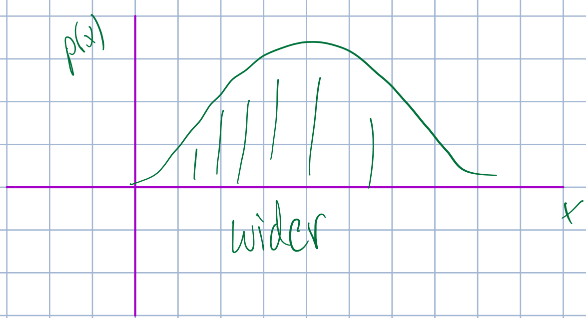
$$\hat{x}\hat{p}\hat{p} = (\hat{p}\hat{x} + i\hbar)\hat{p} = \hat{p}\hat{p}\hat{x} + i\hbar\hat{p} + i\hbar\hat{p} = \hat{p}\hat{p}\hat{x} + 2i\hbar\hat{p}$$

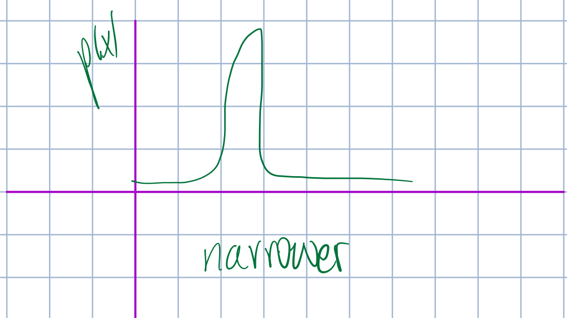
$[\hat{x},\hat{H}] = \frac{1}{2m}(2i\hbar\hat{p}) = -\frac{i\hbar}{m}\hat{p}$ $\quad$ so $\quad \frac{d}{dt}\langle\psi(t)|\hat{x}|\psi(t)\rangle = \frac{1}{m}\langle\psi(t)|\hat{p}|\psi(t)\rangle$

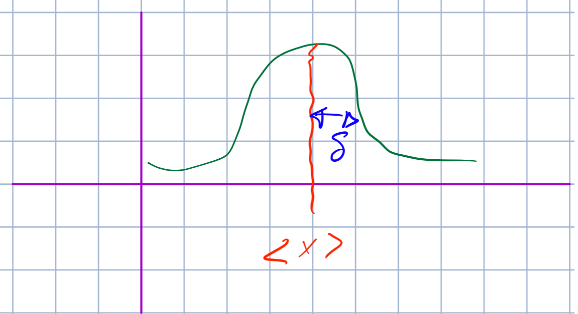
p=mv in avg sense

Check $[\hat{p},\hat{H}] = -i\hbar\frac{dV}{dx}$ $\quad \frac{d}{dt}\langle\psi(t)|\hat{p}|\psi(t)\rangle = \langle\psi(t)| -\frac{dV}{dx} |\psi(t)\rangle$

$[\hat{x},\hat{p}] = -i\hbar$







want to see $x - \delta$

or $x - \Delta x$

but this would cancel out on either side, so we square it

just some #

if A is what you measure w/ some probability $\langle A\rangle$ for its avg

dispersion of probability:
or uncertainty

$$(\Delta A)^2 = \langle (A - \langle A\rangle)^2 \rangle$$

$$= \langle A^2 \underbrace{- 2\langle A\rangle}_{\to -\langle A\rangle^2} + \langle A\rangle^2 \rangle$$

$$= \langle A^2\rangle - \langle A\rangle^2$$

avg. prob.

in QM,

$\langle \psi(t) | \hat{O} | \psi(t) \rangle$ = avg value of $\hat{O}$

$\langle \psi(t) | \hat{O}^2 | \psi(t) \rangle$ = avg value of $\hat{O}^2$

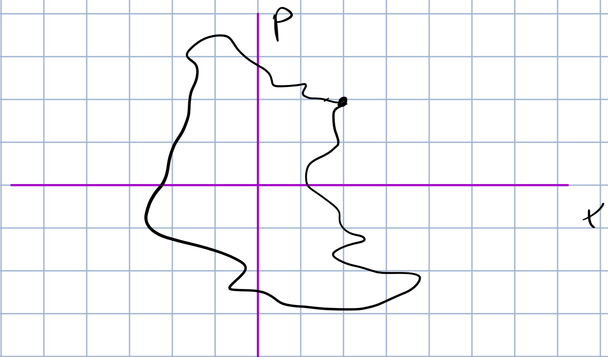
→ avg of x^2 avg of prob. x^2

$$(\Delta x)^2 = \langle \psi(t) | \hat{x}^2 | \psi(t) \rangle - \langle \psi(t) | \hat{x} | \psi(t) \rangle^2$$

$$(\Delta p)^2 = \langle \psi(t) | \hat{p}^2 | \psi(t) \rangle - \langle \psi(t) | \hat{p} | \psi(t) \rangle^2$$

proof of $\Delta x \, \Delta p \geq \hbar/2$ for any normalized $\psi(x,t)$

classical mechanics: phase space (x,p)



in QM: phase space is chopped up