Find Boas

2 goals:
① seperable DEs
② seperable DEs $=0$ w/ coefficients

last time: $y'' + \omega^2 y = 0$ try $y = \sum_{n=0}^{\infty} c_n x^n$

gave recurrence relation: $C_{n+2}(n+2)(n+1) + \omega^2 c_n = 0$

$$C_{n+2} = \frac{-\omega^2 c_n}{(n+1)(n+2)}$$

$$C_{0+2} = \frac{-\omega^2 C_0}{(0+1)(0+2)} = \frac{-\omega^2 C_0}{2} = C_2$$

$$C_{1+2} = \frac{-\omega^2 C_1}{(1+1)(1+2)} = \frac{-\omega^2 C_1}{6} = C_3$$

$$C_{2+2} = \frac{-\omega^2 C_2}{(2+1)(2+2)} = \frac{-\omega^2}{12} C_2 = (-\omega^2)^2 \frac{C_0}{4!} = C_4$$

$$C_5 = \quad . \quad . \quad . \quad = (-\omega^2)^2 \frac{C_0}{5!} = C_1$$

$$C_n \begin{cases} n \text{ even} & (-\omega^2)^{n/2} \frac{C_0}{n!} \\ n \text{ odd} & (-\omega^2)^{\frac{n-1}{2}} \frac{C_1}{n!} \end{cases}$$

flips, one positive one negative

$$y(x) = C_0 \sum_{n \text{ even}} (-\omega^2)^{\frac{n}{2}} \frac{x^n}{n!} \rightarrow C_0 \sum_{n \text{ even}} (-1)^{\frac{n}{2}} \frac{(\omega x)^n}{n!}$$

$$+ C_1 \sum_{n \text{ odd}} (-\omega^2)^{\frac{n-1}{2}} \frac{x^n}{n!} \rightarrow \frac{C_1}{\omega} \sum_{n \text{ odd}} (-1)^{\frac{n-1}{2}} \frac{(\omega x)^n}{n!}$$

$$= C_0 \cos(\omega x) + \frac{C_1}{\omega} \sin(\omega x)$$

Legendre's Eq^n

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notice $\nabla^2 \varphi(x \equiv \cos\theta) = \frac{1}{r^2}\left[(1-x^2)\frac{d^2\varphi}{dx^2} - 2x\frac{d\varphi}{dx}\right]$

Legendres eq^n : $(1-x^2)y''(x) - 2xy'(x) + \underbrace{\overset{\text{mystery}}{\boxed{\ell(\ell+1)}}y}_{\text{constant}} = 0$

invent new infinite series sum f^n's

try $y = \sum_{n=0}^{\infty} c_n x^n$

$\frac{d}{dx}$ $\frac{d}{dx}$ $\frac{d}{dx}$

$$0 = \sum_{n=0}^{\infty}\left[(1-x^2)c_n n(n-1)x^{n-2} - 2xn\,x^{n-1} + \ell(\ell+1)\right]$$

$$= \sum_{n=0}^{\infty}\left[n(n-1)c_n x^{n-2} - \left(n(n-1)c_n + 2nc_n - \ell(\ell+1)c_n\right)x^n\right]$$

when $n=0,1$, sum is zero, can shift $m=n-2$ $m+2=n$

$$= \sum_{m=0}^{\infty}\left[(m+2)(m+1)c_{m+2}x^m\right] - \sum_{n=0}^{\infty}\left[\left(\underbrace{n(n-1)+2n}_{(n+1)n} - \ell(\ell+1)\right)c_n x^n\right]$$

dummy variable

combine

$$= \sum_{m=0}^{\infty}\underbrace{\left[(m+2)(m+1)c_{m+2} - (m(m+1) - \ell(\ell+1))c_m\right]}_{\text{must}=0}x^m$$

Recursion: $c_{m+2} = \frac{m(m+1) - \ell(1+\ell)}{(m+2)(m+1)}c_m$

series converges for $|x| < 1$

if ℓ = integer ≥ 0 $\rightarrow$ $c_{\ell+2} = 0$ series stop

Defines some new f^n's $\rightarrow$ Legendre Polynomials $P_\ell(x=\cos\theta)$

$P_0(x) = 1$ $P_1(x) = x$ $P_2(x) = \frac{3x^2-1}{2}$ $P_3 = \frac{5x^3-3x}{2}$

the $P_\ell(x)$ form a complete expansion basis:

any $f(x=\cos\theta) = \sum_{\ell=0}^{\infty} A_\ell P_\ell(x)$

$\ell(\ell+1)\, y(x)$ term results from a "eigenvalue" situation

RHB Ch 17 $\quad \mathcal{L} \equiv (1-x^2)\frac{d^2}{dx^2} - 2x\frac{d}{dx}$

Legendre DE: $\quad \mathcal{L}[y(x)] = -\ell(\ell+1)\, y(x)$

if sol'n's of $\uparrow$ form an expansion basis $F_\ell(x)$

$$f(x) = \sum_{\ell=0}^{\infty} A_\ell F_\ell(x)$$

then sol'n's of DE $\mathcal{L}[y(x)] = f(x) = \sum_{\ell=0}^{\infty} A_\ell F(x)$

Orthogonality $\quad \int_{-1}^{1} dx\, P_m(x) \cdot \overbrace{P_n(x)}^{\text{Legendre}} \propto \underbrace{\delta_{mn}}_{\text{Kronecker Delta}} = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases}$

compare to $\int_0^{2\pi} \cos(m\theta)\sin(n\theta)\, d\theta$

$\sin(m\theta)\sin(n\theta)$

$\cos(m\theta)\cos(n\theta)$