Last lecture: $y''(x) + A y'(x) + B y(x) = 0$ homogeneous

What if e^{-ax} e^{-bx} must satisfy $a = -b$ $a^2 < 0$

oscillator eqn: $y'' + \omega^2 y = 0 = (D+a)(D+b) = y'' + (a+b)y' + aby$

ω^2 constant $\in \mathbb{R}$

$A \,\&\, B \neq A \,\&\, B$

$a = -b \quad a^2 < 0 \longrightarrow a = \pm i\omega$ 2 solutions

$$(D + i\omega)(D - i\omega) y = (D^2 - i^2\omega^2) y = 0$$

$\longrightarrow y_g(t) = A e^{i\omega t} + B e^{i\omega t}$ general solution

y is position in Hook's Law

$\alpha \,\&\, \beta \in \mathbb{R}$

$A = \alpha_r + \alpha_i i$
$B = \beta_r + \beta_i i$

euler: $e^{\pm i\omega t} = \cos(\omega t) \pm i\sin(\omega t)$

$$y_g(t) = (\alpha_r + \alpha_i i)\cdot(\cos(\omega t) + i\sin(\omega t)) + (\beta_r + \beta_i i)\cdot(\cos(\omega t) - i\sin(\omega t))$$

$$= \alpha_r\cos(\omega t) + \alpha_i i i \sin(\omega t)$$
$$+ \alpha_i i \cos(\omega t) + \alpha_r i \sin(\omega t)$$
$$+ \beta_r \cos(\omega t) - \beta_i i i \sin(\omega t)$$
$$+ \beta_i i \cos(\omega t) - \beta_r i \sin(\omega t)$$

$$= \alpha_r\cos(\omega t) - \alpha_i \sin(\omega t) + \beta_r\cos(\omega t) + \beta_i \sin(\omega t)$$
$$+ i(\alpha_i\cos(\omega t) + \alpha_r\sin(\omega t) + \beta_i\cos(\omega t) - \beta_r\sin(\omega t))$$

$$= \cos(\omega t)[\alpha_r + \beta_r] + \sin(\omega t)[\beta_i - \alpha_i] + i(\cos(\omega t)[\alpha_i + \beta_i] + \sin(\omega t)[\alpha_r - \beta_r])$$

physics: Need a real solution, so we need $\ast$ & $\ast$ to equal zero)

$\alpha_i + \beta_i = 0 \qquad \alpha_r - \beta_r = 0$

$\longrightarrow \alpha_i = -\beta_i \qquad \alpha_r = \beta_r$

$A = \alpha_r + \alpha_i i = \beta_r - \beta_i i$
$B = \beta_r + \beta_i i = \beta_r + \beta_i i$ $\Rightarrow$ conjugates

$$y_g(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t)$$

initial conditions: $y(0) = C_1$
$\dot{y}(0) = \omega C_2$

Non homogeneous!

15.1.2

Some 2nd order ODE:

$$\underbrace{a y''}_{\text{acceleration}} + \underbrace{b y'}_{\text{damping}} + \underbrace{c\, y(x)}_{\text{restoring force}} = \underbrace{f(x)}_{\text{driving force}}$$

say a driven, damped oscillator

we already now homogeneous gets exp( ) solutions

if $f(x)$ is some exp( ), we can expand a fourier series & add up

$$\text{if } f(x) = \overset{\text{constant}}{D}\, e^{dx}$$

$\exp(\star \cdot x) \rightarrow$ indpt. fns, noway add up different types exponentials & end up w/ e^{dx}, so need same form

need same exponential constant

$$\Rightarrow \quad y(x) = A e^{dx}$$

$$\Rightarrow \quad A e^{dx}\left[a d^2 + b d + c\right] = D e^{dx}$$

$$A = \frac{D}{ad^2 + bd + c}$$

now know amplitude

A, D could be complex

Fourier expand for general case: what if $f(x) = D e^{dx} + \overset{\text{constants}}{\Delta}\, e^{\delta x}$

Linear means $\rightarrow$ $y = y_D + y_\Delta = A e^{dx} + B e^{\delta x}$ works

$$A = \frac{D}{ad^2 + bd + c} \qquad B = \frac{\Delta}{a\delta^2 + b\delta + c}$$

break down to single one component problem

general $y_g(x) = \underbrace{y_p(x)}_{\text{driving}} + \underbrace{y_c(x)}_{\text{transient}}$, $\quad a y_c'' + b y_c' + c y_c = 0$

Driven Oscillator

$$y_c = G e^{\gamma x} \quad \longrightarrow \quad (a\gamma^2 + b\gamma + c) = 0$$

$$\gamma = -\frac{\gamma}{2a} \pm \frac{1}{2}\sqrt{\left(\frac{b}{a}\right)^2 - 4\left(\frac{a}{c}\right)}$$ Quadratic eq^{n}

$$y_g(x) = \overbrace{\frac{D}{ad^2+bd+c}\, e^{dx}}^{y_p(x)} + \overbrace{e^{-\frac{b}{2a}x}\left[G_+ e^{\frac{1}{2}\sqrt{\ }} + G_- e^{-\frac{1}{2}\sqrt{\ }}\right.}^{y_c(x)}$$

Problem case:

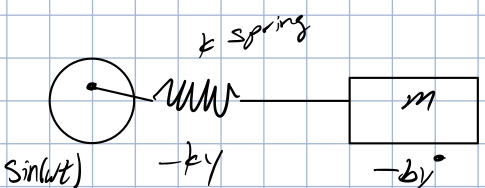
$$y'' - 2ay' + a^2 y = e^{ax} \quad \longrightarrow \quad y_p = Ae^{ax}$$

$$\underbrace{(a^2 - 2a^2 + a^2)}_{0} A = Ae^{ax}$$

$$\vdots$$

$$\longrightarrow \quad y_p(x) = Ax^2 e^{ax}$$

Damped Driven Oscillator



f

$$F = m\ddot{y} = -ky - b\dot{y} + \overbrace{F_{Drive}(t)}^{D\sin(\omega t)}$$

$$\ddot{y} + \underbrace{\frac{b}{m}}_{\gamma}\dot{y} + \underbrace{\frac{k}{m}}_{\omega_0^2} y = \overbrace{\frac{D}{m}\sin(\omega t)}^{\text{Real of } -ie^{i\omega t}}$$

sin $\longrightarrow e^{i\omega t}$, take $Re[\quad]$ like before

$$y_p = Ae^{i\omega t}$$

$$\left[(i\omega)^2 + \gamma(i\omega) + \omega_0^2\right] Ae^{i\omega t} = -i\frac{D}{m} e^{i\omega t}$$

$$A = \frac{-i\, D/M}{\omega_0^2 + (i\omega)^2 + i\gamma\omega} = \frac{-i\, D/M}{(\omega_0^2 - \omega^2) + i\gamma\omega}$$

we only want real part, so let's multiply by complex conjugate

$$A = \frac{-i\, D/\mu}{(\omega_0^2-\omega^2)+i\gamma\omega} \cdot \frac{(\omega_0^2-\omega^2)-i\gamma\omega}{(\omega_0^2-\omega^2)-i\gamma\omega} = \frac{D}{\mu} \cdot \frac{\gamma\omega - i(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$$

$$y_p(t) = \mathrm{Re}\left[A e^{i\omega t}\right] = \mathrm{Re}\left[\frac{D}{\mu} \cdot \frac{1}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2} \cdot [\gamma\omega - i(\omega_0^2-\omega^2)] \cdot e^{i\omega t}\right]$$

$$= \frac{D}{\mu} \cdot \frac{1}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2} \cdot \mathrm{Re}\left[[\gamma\omega - i(\omega_0^2-\omega^2)] \cdot \overbrace{e^{i\omega t}}^{\text{euler}}\right]$$

$$= \quad \cdots \quad \cdot \mathrm{Re}\left[[\gamma\omega - i(\omega_0^2-\omega^2)] \cdot (\cos(\omega t) + i\sin(\omega t))\right]$$

$$y_p(t) = \frac{D}{\mu} \, \frac{\gamma\omega\cos(\omega t) + (\omega_0^2-\omega^2)\sin(\omega t)}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$$

$\longrightarrow$ resonance denominator

$y_c(t)$:

$$\ddot{y}_c + \gamma \dot{y}_c + \omega_0^2 y_c = 0 \qquad y_c \propto e^{\alpha t}$$

$$(\alpha^2 + \gamma\alpha + \omega_0^2)\, e^{\alpha t} = 0$$

$$\alpha^2 + \gamma\alpha + \omega_0^2 = 0$$

$$\longrightarrow \alpha = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$$

for $\gamma \ll \omega_0$

$$y_c(t) = \left[\underline{\underline{A_+}}\, e^{i\sqrt{\omega_0^2 - \gamma^2/4}} + \underline{\underline{A_-}}\, e^{-i\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}}\right] e^{-\gamma/2\, t}$$

transient term

$$y_g(x) = y_p(t) + \mathrm{Re}[y_c(t)]$$

$$(\gamma\omega - i(\omega_0^2-\omega^2))\,(\cos(\omega t) + i\sin(\omega t))$$

$$\gamma\omega\cos(\omega t) - i^2(\omega_0^2-\omega^2)\sin(\omega t) + \gamma\omega i\sin(\omega t) - i(\omega_0^2-\omega^2)\cos(\omega t)$$

$$\gamma\omega\cos(\omega t) + (\omega_0^2-\omega^2)\sin(\omega t) + i\left[\gamma\omega\sin(\omega t) - (\omega_0^2-\omega^2)\cos(\omega t)\right]$$

$$\mathrm{Re}[\ \text{''}\ \text{''}\ ] = \gamma\omega\cos(\omega t) + (\omega_0^2-\omega^2)\sin(\omega t)$$