Expansion:

$$|w\rangle = \sum_i \underbrace{\langle i|w\rangle}_{w_i\text{, a scalar}} |i\rangle \qquad \text{"ket"}$$

$$\langle v| = \sum_j \underbrace{\langle v|j\rangle}_{v_j^*} \langle j|$$

$\longrightarrow$ b/c $\langle j|v\rangle = \langle v|j\rangle^*$ postulate!
so that $\langle v|v\rangle \geq 0$ & real

Notice $\langle v|w\rangle = \sum_i \sum_j v_j^* w_i \langle j|i\rangle = \sum_i v_i^* w_i$ for $\langle j|i\rangle = \delta_{ij}$

Now let an operator act $\mathcal{L}|w\rangle = |w'\rangle$

Then $\langle v|w'\rangle = \langle v|\mathcal{L}|w\rangle = \left(\sum_i v_i^* \langle i|\right)\left(\mathcal{L} \sum_j w_j |j\rangle\right)$

$$= \sum_i \sum_j v_i^* w_j \underbrace{\langle i|\mathcal{L}|j\rangle}_{\mathcal{L}_{ij}} \text{ - operator matrix}$$

instead of transforming w & getting w', lets transform the bases $\mathcal{L}_{ij}$

can we transform a bra?

Q: can I formulate as $\langle v'|w\rangle = \langle v|w'\rangle$?

we know $\langle v'|w\rangle = \langle w|v'\rangle^* = \left[\sum_j \underbrace{\langle w|j\rangle}_{w_j^*} \langle j| \tilde{\mathcal{L}} \sum_i \underbrace{\langle i|v\rangle}_{v_i} |i\rangle\right]^*$ ① ②

① we know how $\mathcal{L}|v\rangle$ works, so have move bra to right & * $= \sum_i \sum_j v_i^* w_j \underbrace{\langle j|\tilde{\mathcal{L}}|i\rangle^*}_{\tilde{\mathcal{L}}_{ji}^*}$ ← ③

② expand & have some transformation $\tilde{\mathcal{L}}$ different from last example

③ distribute *

$$\tilde{\mathcal{L}}_{ji}^* = \mathcal{L}_{ij} \longrightarrow \tilde{\mathcal{L}} = (\mathcal{L}^T)^* = \mathcal{L}^\dagger$$ ← hermitian adjoint

$$\Longrightarrow \langle v|\mathcal{L}|w\rangle = \langle w|\mathcal{L}^\dagger|v\rangle^*$$

Apply to Basis transforms RHB 8.15, 21.1, 21.2

new basis — operator on old basis

let $|i'\rangle = \Omega|i\rangle$

$\rightarrow \langle j'| = \langle j|\Omega^{+}$

basis shouldn't matter for physics

$\rightarrow \langle j'|i'\rangle = \langle j|\Omega^{+}\Omega|i\rangle = [\Omega^{+}\Omega]_{ji}$

if $\Omega^{+}\Omega = \underset{\text{identity matrix}}{1}$ then inner product is basis indpt.

$\rightarrow \Omega^{+} = \Omega^{-1}$ "unitary operator"

construct Ω explicitly:

$N=2 \rightarrow (i,j)$ — dot product, then act in specific old basis — expand into old basis

$|i'\rangle = \sum_{k=1}^{N} \langle k|i'\rangle |k\rangle = \Omega|i\rangle$

$\rightarrow$ expanding object in new basis in terms of old basis

$\langle j|i'\rangle = \sum_k \langle k|i'\rangle\langle j|k\rangle = \langle j|\Omega|i\rangle \equiv \Omega_{ji}$

if $\langle j|k\rangle = \delta_{jk} \rightarrow \Omega_{ji} = \langle j|i'\rangle$

flip

AND $\Omega_{ji}^{*} = \langle i'|j\rangle = \langle j|i'\rangle^{*}$ is inverse transform

$|i\rangle = \Omega^{-1}|i'\rangle = \sum_k \langle k'|i\rangle |k'\rangle$

$\rightarrow$ expand in terms of new basis

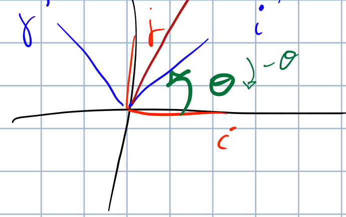
$\rightarrow$ old basis in new basis

Summarize:

$\langle i|j'\rangle = \Omega_{ij}$ gives $|i'\rangle$ from $|i\rangle$

$\langle i'|j\rangle = \Omega^{-1}_{ij}$ gives $|i\rangle$ from $|i'\rangle = \Omega_{ij}^{+}$

example: rotation in 2D



$$\mathcal{R}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

rotate back

$$\mathcal{R}^{-1}(\theta) = \mathcal{R}(-\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \mathcal{R}^T(\theta) \rightarrow \mathcal{R} \text{ is unitary}$$

mirror: $T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = T^T = T^{-1}$ funny matrix

---

How does operator matrix change on basis change?

$\mathcal{L}_{ij} = \langle i | \mathcal{L} | j \rangle$ change basis

m & n are old basis like previous k

$$\mathcal{L}_{ij}' = \langle i' | \mathcal{L} | j' \rangle = \sum_m \langle i' | m \rangle \langle m | \mathcal{L} \sum_n \langle n | j' \rangle | n \rangle$$

$$= \sum_m \sum_n \underbrace{\langle i' | m \rangle}_{\Omega^+_{im}} \underbrace{\langle n | j' \rangle}_{\Omega_{nj}} \underbrace{\langle m | \mathcal{L} | n \rangle}_{\mathcal{L}'_{mn}}$$

matrix of all scalars over $\sum_{m,n}$ for $\langle i |$

matrix of operator in old basis

Ω can be a matrix

↳ unprimed prime

↳ prime → unprime (what's projection of j' on old n)

This is just matrix multiplication

if Ω real

$$\mathcal{L}_{ij}' = \sum_m \sum_n \Omega^T_{im} \Omega_{nj} \mathcal{L}_{mn}$$

$$= \sum_m \Omega^T_{im} \underbrace{\left[\sum_n \Omega_{nj} \mathcal{L}_{mn} \right]}_{\sum_n \mathcal{L}_{mn} \Omega_{nj}} = [\mathcal{L}\,\Omega]_{mj}$$

$$\mathcal{L}_{ij}' = [\Omega^T \mathcal{L}\, \Omega]_{ij}$$

$$\underline{\mathcal{L}'} = \underline{\Omega^T}\ \underline{\mathcal{L}}\ \underline{\Omega}$$

Things that transform like this way are tensors