Last time: coupled DE $\quad \frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -\alpha & \beta \\ \alpha & -\beta \end{bmatrix}}_{\equiv A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

① Get eigenvalues & eigenvectors of A:

$$\lambda_1 = -(\alpha+\beta) \qquad \vec{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$\lambda_2 = 0 \qquad \vec{v}_2 = \frac{1}{\sqrt{\alpha^2+\beta^2}}\begin{bmatrix} \beta \\ \alpha \end{bmatrix} = \begin{bmatrix} \beta/r \\ \alpha/r \end{bmatrix}$$

② Create Matrices

$$V \equiv \begin{array}{c} \begin{bmatrix} 1/\sqrt{2} & \beta/r \\ -1/\sqrt{2} & \alpha/r \end{bmatrix} \\ \uparrow \qquad\quad \uparrow \\ \vec{v}_1 \qquad\quad \vec{v}_2 \end{array} \quad \& \quad D \equiv \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -(\alpha+\beta) & 0 \\ 0 & 0 \end{bmatrix}$$

③ Notice: $\quad A \cdot V = V \cdot D \rightarrow V^{-1} A V = D \quad$ def. new basis

④ New basis $\quad \frac{d}{dt}\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = D \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} \quad$ decoupled eq's

' prime - new basis

$$\dot{x}_1' = \lambda_1 x_1' \rightarrow x_1' = c_1 e^{\lambda_1 t} = c_1 e^{-(\alpha+\beta)t}$$

$$\dot{x}_2' = \lambda_2 x_2' \rightarrow x_2' = c_2 e^{\lambda_2 t} = c_2$$

⑤ Transform back to old basis

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = V \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & \beta/r \\ -1/\sqrt{2} & \alpha/r \end{bmatrix} \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{c_1}{\sqrt{2}} e^{\lambda_1 t} + \frac{\beta c_2}{r} \\ -\frac{c_1}{\sqrt{2}} e^{\lambda_1 t} + \frac{\alpha c_2}{r} \end{bmatrix}$$

$$A \cdot \vec{x} = \lambda \vec{x}$$

$$V^{-1} A (V V^{-1}) \vec{x} = \lambda V^{-1} \vec{x}$$

$$D \underbrace{(V^{-1} \vec{x})}_{\vec{x}'} = \lambda (V^{-1} \vec{x})$$

Check algebra $V^{-1} A V$

get $V^{-1} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$V^{-1} V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & \beta/r \\ -1/\sqrt{2} & \alpha/r \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

RHB 17.1

Fourier Expansion = Basis expansion

$|f(x)\rangle$ could be a valid LVS

w/ inner product defined as $\langle f|g\rangle = \int_{-0}^{0} f^*(x) g(x)\, dx$

limits of x in LVS

Define LVS properties:

Periodicity: $f(2\pi) = f(0)$

continuous $\frac{df}{dx}(2\pi) = \frac{d}{dx} f(0)$ $\to$ not used today

define operator $\mathcal{L} = \frac{d^2}{dx^2}$

find a basis using eigenvalues of $\mathcal{L}$:

$$\mathcal{L}\,|y(x)\rangle = \lambda\,|y(x)\rangle \quad \Longleftrightarrow \quad \frac{d^2y}{dx^2} = \lambda y(x)$$

solved by $y = A e^{\sqrt{\lambda} x}$

allowed λ determined by LVS properties

satisfies periodicity & derivative rules

$$y(2\pi) \equiv A e^{2\pi\sqrt{\lambda}} \equiv y(0) = A e^{0} \to e^{2\pi\sqrt{\lambda}} = 1 \to \sqrt{\lambda} = n i = n\sqrt{-1}$$

eigenvectors $\to$ eigenfunctions e^{inx} $\quad n = 0, \pm 1, \pm 2, \ldots$

look @ inner product n & m are bases

$$\langle n | m \rangle = \int_0^{2\pi} \left(e^{inx}\right)^* e^{imx}\, dx = \int_0^{2\pi} e^{-inx} e^{imx}\, dx$$ defined inner product

Basis expansion: $$| f(x) \rangle = \sum_{n=-\infty}^{\infty} \underbrace{\langle n | f(x) \rangle}_{\text{coefficients}} | n \rangle$$ $\rightarrow e^{inx}$, basis fn's

$$\rightarrow \langle n | m \rangle = \frac{1}{(m-n)i}\, e^{(m-n)x} \Big|_0^{2\pi} = \ldots$$

$\rightarrow$ 0 for $m \neq n$ denominator doesn't blow up $e^{\text{integer} \cdot 2\pi} - e^{\text{integer} \cdot 0} \rightarrow 1 - 1$

$\rightarrow$ for $m=n$, use euler formula:

$$= \int_0^{2\pi} (\cos(nx) - i\sin(nx))(\cos(mx) + i\sin(mx))\, dx$$

$$= \left[\int \cos(nx)\cos(mx)\, dx + \int \sin(nx)\sin(mx)\, dx \right] + i \left[\int \cos(nx)\sin(mx) - \int \sin(nx)\cos(mx)\, dx \right]$$

$$= \pi \delta_{mn} + \pi \delta_{mn} + 0 - 0$$

$$= 2\pi \delta_{mn}$$

Get familiar w/ Fourier Series

$$| f(x) \rangle = \sum_{n=-\infty}^{\infty} \langle n | f \rangle | n \rangle = \sum_{n=-\infty}^{\infty} C_n e^{inx} \equiv \mathbb{R}$$

another condition $\rightarrow$ C_n are complex

$$= \sum_{-\infty}^{-1} C_n e^{inx} + \sum_{1}^{\infty} C_n e^{inx} + C_0$$

dummy index $n \rightarrow -n$

$$= \sum_{\infty}^{1} C_{-n} e^{-inx} + \sum_{1}^{\infty} C_n e^{inx} + C_0$$

$\hookrightarrow$ order of sum doesn't matter $\sum_{\infty}^{1} = \sum_{1}^{\infty}$

$$= \sum_{n=1}^{\infty} \left[C_{-n} \cdot (\cos(nx) - i\sin(nx)) + C_n (\cos(nx) + i\sin(nx)) \right] + C_0$$

$$= C_0 + \sum_{n=1}^{\infty} \left[(C_{-n} + C_n) \cos(nx) + i (C_n - C_{-n}) \sin(nx) \right]$$

restrict $(C_{-n} + C_n)$ & $i(C_n - C_{-n})$ to be real

$\rightarrow$ done by defining $C_{-n} = C_n^*$

$\rightarrow C_n + C_{-n} = C_n + C_n^* = 2\,\mathrm{Re}[C_n] = A_n$

$i(C_n - C_{-n}) = i(C_n + C_n^*) = 2i\,\mathrm{Im}[C_n] = B_n$

$\rightarrow |f(x)\rangle = C_0 + \sum_{n=1}^{\infty}[A_n \cos(nx) + B_n \sin(nx)]$

Properties of "Special Operators"

$'' \rightarrow \frac{d^2}{dx^2}$

consider $\langle f|\mathcal{L}|g\rangle = \int_0^{2\pi} dx\, f^* g''$ for $\mathcal{L} = \frac{d^2}{dx^2}$

$= \int dg'\, f^*$

integrate by parts $= \underbrace{f^*(x)\, g'(x)\Big|_0^{2\pi}}_{BC \rightarrow 0} - \int g'\, df^*$

$\rightarrow -\int f^{*\prime}\, dg$

integrate by parts $= -[\underbrace{f^{*\prime} g\Big|_0^{2\pi}}_{=0 \text{ b/c BC}} - \int g\, df^{*\prime}]$

$= \int g\, df^{*\prime}$

$\langle f|\mathcal{L}|g\rangle = +\int g f^{*\prime\prime} dx$

$= [\int g^* f'' dx]^*$

$= \langle g|\mathcal{L}|f\rangle^*$

expand in basis

expand g

expand f

$= \underbrace{\langle g|i\rangle^* \quad \langle i|\mathcal{L}|\langle j|f\rangle|j\rangle^*}_{\text{coefficients}} =$

$= \langle g|i\rangle^* \langle j|f\rangle^* \underbrace{\langle i|\mathcal{L}|j\rangle^*}_{L_{ij}^*}$

1

also

$= \langle f|j\rangle \langle j | \mathcal{L} | \langle i | g\rangle | i\rangle$

$= \langle f|j\rangle \; \langle i|g\rangle \; \langle \mathcal{L}_{ij}\rangle$ 2

now $\langle g|i\rangle^* = \langle i|g\rangle$

2 $\langle f|j\rangle \langle i|g\rangle \mathcal{L}_{ij} = \langle i|g\rangle \; \langle f|j\rangle \mathcal{L}_{ij}^*$ 1

$\mathcal{L}_{ij} = \mathcal{L}_{ij}^* \longleftrightarrow \underline{\mathcal{L}^\dagger} = \underline{\mathcal{L}}$ hermitian