Linear Operators

$\mathcal{L}|a\rangle = |b\rangle$

what if this is $|x(t)\rangle$ & $\mathcal{L} = \frac{d}{dt}$?

consider $x(t) \to x(t_0), x(t_1), x(t_2), \ldots \to \infty$

$\to$ so inner product $\langle f(t)|g(t)\rangle \to f^*(t_0)g(t_0) + \ldots + f^*(t_n)g(t_n)$

$\to \int_0^\infty f^*(t)g(t)\,dt$

Basis expansion: $|a\rangle = \sum_i \langle i|a\rangle \; |i\rangle$

then for linear $\mathcal{L}$: $\mathcal{L}|a\rangle = \sum_i \mathcal{L}\{\sum \langle i|a\rangle |i\rangle\}$

$= \sum_i \underbrace{\langle i|a\rangle}_{a_i} \mathcal{L}|i\rangle$

Match w/ expansion of $|b\rangle = \sum_j \langle j|b\rangle \; |j\rangle$

$b_j = \langle j|b\rangle = \langle j|\{\overbrace{\sum_i a_i}^{\text{move out}} \mathcal{L}|i\rangle\} = \sum_i a_i \langle j|\mathcal{L}|i\rangle$

$\to$ so collect $\langle j|\mathcal{L}|i\rangle$

Example: $\mathcal{L} = \frac{d}{dt}$, $f' = \lambda f \to f = e^{\lambda t} \to |i\rangle = e^{it}$

Another Example: $\mathcal{L} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad |a\rangle = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

choose basis: $|1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad |2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\mathcal{L}|1\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0\cdot 1 \\ 1\cdot 1 \end{bmatrix}$ $\quad \mathcal{L}|2\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

dot product

$\langle 1|\mathcal{L}|1\rangle = \mathcal{L}_{11} = \overbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}^{1^T} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$ $\quad 0\cdot 1 + 0\cdot 1$

$\langle 1|\mathcal{L}|2\rangle = \mathcal{L}_{12} = \overbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}^{1^T} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1$ $\quad 1\cdot 1 + 0\cdot 0$ $\quad - 1$

1 in basis 1

$1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\langle 2|\mathcal{L}|1\rangle = \mathcal{L}_{21} = \overbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}^{2^T} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1$ $\quad 0\cdot 0 + 1\cdot 1$ $\quad - 2$

$2^T$

$\langle 2 | \mathcal{L} | 2 \rangle = \mathcal{L}_{22} = [\,0 \;\; 1\,] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0 \quad [0\cdot 1 + 1 \cdot 0]$ $\mathcal{L}$ in basis 2

eigenvalues: $\mathcal{L}|a\rangle = \lambda |a\rangle \rightarrow \det(\mathcal{L} - \lambda \mathbb{1}) = 0 = \det\begin{pmatrix} 0-\lambda & 1 \\ 1 & 0-\lambda \end{pmatrix}$

$-\lambda \cdot -\lambda - 1 \cdot 1 = \lambda^2 = 0 \rightarrow \quad \lambda = \pm 1$

$[\mathcal{L} - \lambda \mathbb{1}] \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$ ↳ eigenvalue

eigenvectors: $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ would have made $\mathcal{L}_{ij}$ diagonal

$\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v_1 \begin{bmatrix} -\lambda \\ 1 \end{bmatrix} + v_2 \begin{bmatrix} 1 \\ -\lambda \end{bmatrix} = \begin{bmatrix} -\lambda v_1 + v_2 \\ v_1 - \lambda v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad v_2 = \lambda v_1, \; v_1 = \lambda v_2 \quad \lambda_1 \rightarrow v_2 = v_1 \rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \lambda_{-1} \rightarrow v_1 = -v_2 \rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Using basis $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}}$ & $\begin{bmatrix} 1 \\ -1 \end{bmatrix} \frac{1}{\sqrt{2}}$ : $\equiv |1'\rangle, |2'\rangle$

$\mathcal{L}|1'\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ldots$

$\mathcal{L} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \left[ 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right] = \frac{1}{\sqrt{2}} \left[ \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$* \; \mathcal{L}_{11}' = \langle 1' | \mathcal{L} | 1' \rangle = \frac{1}{\sqrt{2}} \cdot [1 \;\; 1] \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1$

$\frac{1}{2} [1 \;\; 1] \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2}(1 \cdot 1 + 1 \cdot 1) = \frac{1}{2}(2) = 1$ dot product

$* \; \mathcal{L}_{12}' = 0$

$* \; \mathcal{L}_{21}' = 0$

$* \; \mathcal{L}_{22}' = -1$

$\mathcal{L}' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

diagonal! :)

① getting new basis
② putting them thru $\mathcal{L}$ transformation
③ seeing $\mathcal{L}$ in new basis <new/new transformed>

in new basis, vector in new basis going through newly translated $\mathcal{L}$ can be described as new vector in new basis

$\mathcal{L}'|a'\rangle = |b'\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a_1' \\ a_2' \end{bmatrix} = \begin{bmatrix} a_1' \\ -a_2' \end{bmatrix} = \begin{bmatrix} b_1' \\ b_2' \end{bmatrix}$

in general, if operator has eigenvalues: eigenvectors are orthogonal

& $\langle i | \mathcal{L} | j \rangle$ is diagonal in this basis $= \begin{bmatrix} \lambda_1 & & 0 \\ 0 & \lambda_2 & \\ & & \lambda_3 \ldots \end{bmatrix}$

Example, For 2x2 Matrix operator $A$, $A|v_n\rangle = \lambda_n |v_n\rangle$

Notice: $A \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} = A \cdot \begin{bmatrix} v_1^1 & v_2^1 \\ v_1^2 & v_2^2 \end{bmatrix}$ transformation matrix $= V$

$\nearrow$ 2 $\rightarrow$ not exp., second element of v

$A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix}$

$\rightarrow \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} \cdot \begin{bmatrix} v_1^1 & v_2^1 \\ v_1^2 & v_2^2 \end{bmatrix} = \begin{bmatrix} A_{11} v_1^1 + A_{21} \cdot v_1^2 & \ldots \\ A_{12} v_1^1 + A_{22} v_1^2 & \ldots \end{bmatrix}$

$\rightarrow v_1^1 \begin{bmatrix} A_{11} \\ A_{12} \end{bmatrix} + v_1^2 \begin{bmatrix} A_{21} \\ A_{22} \end{bmatrix} + v_2^1 \begin{bmatrix} A_{11} \\ A_{12} \end{bmatrix} + v_2^2 \begin{bmatrix} A_{21} \\ A_{22} \end{bmatrix}$

$\rightarrow [A \cdot V]_{\text{column } n, \text{ row } i} = \sum_k A_{ik} [\vec{v}_n]_k$ $\quad \rightarrow$ nth eigenvector

$= [A \cdot \vec{v}_n]_i = \lambda_n [\vec{v}_n]_i$

① $A$ in some general basis
② Find its eigenvalues
③ Change basis to make

$$\longrightarrow \begin{cases} A_{11} v_1^1 + v_1^2 A_{21} & A_{11} v_2^1 + A_{21} v_2^2 \\ A_{12} v_1^1 + v_1^2 A_{22} & A_{12} v_2^1 + A_{22} v_2^2 \end{cases}$$

diagonal matrix of eigenvalues

$$\underline{A} \cdot \underline{V} = \underline{V} \cdot \underline{D}$$

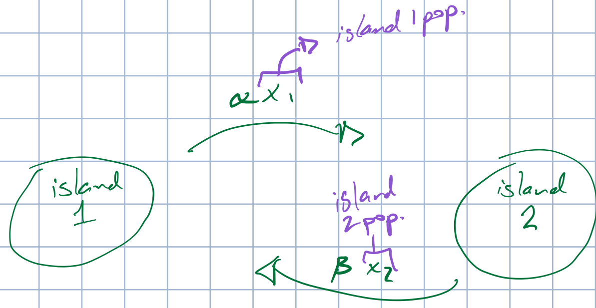
$$D = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} \longrightarrow \text{could be } \langle i' | A | j' \rangle$$

1) diagonal
Get its eigen values / vectors
2) Solve w/ diagonal matrix $\underline{D}$
3) Go back to old basis
Use diagonal of eigenvectors $\underline{V}$

old basis $|i\rangle = V \; |i'\rangle$ new basis

change basis so $A$ is diagonal

## Practical example




world w/ 2 islands, people migrate from 1 to another, vice versa

every year $\alpha x_1$ go $1 \to 2$

" " $\beta x_2$ go $2 \to 1$

$$\left.\begin{aligned} \dot{x}_1(t) &= -\alpha x_1 + \beta x_2 \\ \dot{x}_2(t) &= -\beta x_2 + \alpha x_1 \end{aligned}\right] \text{LVS} \equiv x_{1,2}(t)$$

not diagonal

write as matrix: $\frac{d}{dt}[\vec{x}] = \frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -\alpha & \beta \\ \alpha & -\beta \end{bmatrix}}_{A \text{ from last example}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

not derivative, basis notation

if $A$ were diagonal: $\begin{bmatrix} \dot{x}_1' \\ \dot{x}_2' \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1' \\ \lambda_2 x_2' \end{bmatrix}$

$$x_1'(t) = C_1 e^{\lambda_1 t} \qquad x_2'(t) = C_2 e^{\lambda_2 t}$$

use eigenvalue procedure: $A \vec{v}_n = \lambda_n \vec{v}_n$

goto "diagonal" frame.
Find translation from new to old bases using eigenvectors

" $\det[A - \lambda \mathbb{1}] = \det\begin{bmatrix} -\alpha-\lambda & \beta \\ \alpha & -\beta-\lambda \end{bmatrix} = (\lambda+\alpha)(\lambda+\beta) - \alpha\beta = 0$

$$\longrightarrow \lambda_{\underline{1}} = -(\alpha+\beta) \qquad \lambda_{\underline{2}} = 0$$

eigenvectors:

$$[A - \lambda \mathbb{1}]\vec{v}_{\underline{1}} = 0 = \begin{bmatrix} -\alpha+(\beta+\alpha) & \beta \\ \alpha & -\beta+(\beta+\alpha) \end{bmatrix} \vec{v}_1 = \begin{bmatrix} \beta & \beta \\ \alpha & \alpha \end{bmatrix} \vec{v}_1$$

$$\longrightarrow \vec{v}_1 \propto \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

proportional to

Using

$$[\underline{A} - \lambda \mathbb{1}]\vec{v}_{\underline{2}} = 0 = \begin{bmatrix} -\alpha - 0 & \beta \\ \alpha & -\beta - 0 \end{bmatrix}\vec{v}_2 = \begin{bmatrix} -\alpha & \beta \\ \alpha & -\beta \end{bmatrix}\vec{v}_2 = 0$$

$$\vec{v}_2 \propto \begin{bmatrix} \beta \\ \alpha \end{bmatrix}$$

Transformation Matrix

$$\underline{V} = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & \cancel{\frac{\beta}{\sqrt{\alpha^2+\beta^2}}} \\ -1/\sqrt{2} & \cancel{\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}} \end{bmatrix}$$ ← normalized

$$\vec{x} = \underline{V}\ \vec{x}'$$

→ translates from new to old basis
primed to unprimed

① Get eigenvalues & eigenvectors of $\underline{A}$

② Create Matrices $\underline{V} = [\text{eigenvector 1} \quad \text{eigenvector 2}]$

$x_1, x_2$ → new basis where $A$ is diagonal → $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \lambda \mathbb{1}$

① compute new to old basis
② do transformation

① do transformation → b/c bases are eigenvalues, they just get scaled!
② compute new to old basis

③ Notice $\underline{A} \cdot \underline{V} = \underline{V}\underline{D}$ → $\underbrace{\underline{V}^{-1}\underline{A}\,\underline{V} = \underline{D}}$

④ New basis $x_1', x_2'$ previously: $\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underline{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\frac{d}{dt}\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \underline{D}\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1' \\ \lambda_2 x_2' \end{bmatrix}$$

gives 2 decoupled eq$^{n}$'s

$$\dot{x}_1' = \lambda_1 x_1' \rightarrow x_1' = C_1 e^{\lambda_1 t}$$
$$\dot{x}_2' = \lambda_2 x_2' \rightarrow x_2' = C_2 e^{\lambda_2 t}$$

plug in $\lambda_1$ & $\lambda_2$

⑤ Transform to old basis

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underline{V}\begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$$

$$\underline{A} \cdot \vec{x} = \lambda \vec{x}$$

$$\underline{A} \cdot \underline{V} \cdot \underline{V}^{-1}\vec{x} = \lambda \vec{x}$$ just put it in ($V V^{-1} = \mathbb{1}$)

$$\underbrace{\underline{V}^{-1} \cdot \underline{A} \cdot \underline{V}}_{\underline{D}} \cdot V^{-1}\vec{x} = \lambda\, \underline{V}^{-1}\vec{x}$$ $\underline{V}^{-1}$ both

$$\underline{D}\,\underbrace{\underline{V}^{-1}\vec{x}}_{\vec{x}'} = \lambda\,\underbrace{\underline{V}^{-1}\vec{x}}_{\vec{x}'}$$ def$^{n}$

$$\underline{D}\,\vec{x}' = \lambda\,\vec{x}'$$