Monday noon → Fri noon

PDF

$\underline{M} = M_{ij}$

RHB 8.4 – Properties of Matrices + Index Notation

Last time: Base Expansion, Basis Change, Matrix Basis

"adjoint" consider a 2x2 Matrix operator acting 1x2, 2x1 LVS vectors

$|v\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$ $\underline{M} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$ $\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{21} \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} =$

$a\begin{bmatrix} m_{11} \\ m_{21} \end{bmatrix} + b\begin{bmatrix} m_{12} \\ m_{22} \end{bmatrix}$

$$\underline{M}|v\rangle = \underline{M} \cdot \begin{bmatrix} a \\ b \end{bmatrix} \longrightarrow \left[M\begin{bmatrix} a \\ b \end{bmatrix}\right]_i = \sum_{j=1}^{2} M_{ij} \cdot v_j$$

↳ $M_{i1} \cdot v_1 + M_{i2} \cdot v_2$

↓ change order

$\langle v| M$? $\begin{bmatrix} a \\ b \end{bmatrix} M$ doesn't make sense

→ $\langle v| = [a^* \ b^*]$ ~ BRA (adjoint) KET $|v\rangle$

$*$ → complex conjugate

dagger transpose

$$\langle v| = v^\dagger = [v^T]^*$$

$A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $A^T = [1 \ 2]$

→ transpose

Some basics

$[\underline{M}^*]_{ij} = [M_{ij}]^*$ take complex conjugate of everything in matrix

→ Hermitian conjugate

~> $[\lambda \underline{M}]_{ij} = [\lambda M_{ij}]$ adjoint = Hermitian conjugate same thing!

→ matrix

transpose: $[\underline{M}^T]_{ij} = M_{ji}$ ~> $(\underline{A} \cdot \underline{B})^T = \underline{B}^T \cdot \underline{A}^T$ in that order

$$\sum_{k=1}^{2} A_{jk} B_{ki} = \sum_k [A^T_{kj}] \cdot [B^T_{ik}]$$

$$= \sum_k [B_{ik} A_{jk}]^T$$

For 1x2, 2x1 Matrices: $\vec{v}, \vec{w}$

$$\langle v|w\rangle = \vec{v}^\dagger \cdot \vec{w} = [v_1^* \ v_2^* \dots] \begin{bmatrix} w_1 \\ w_2 \\ \vdots \end{bmatrix} = \sum_{k=1}^{2} v_k^* w_k$$

$|\vec{v}|^2 = \langle v|v\rangle = \sum_{k=1}^{2} v_k^* v_k$ → is real

any complex multiplied by itself is in $\mathbb{R}$

Eigenvalues & Eigenvectors

RHB 8.13

consider column vectors in LVS: $\vec{x} \equiv \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \rightarrow$ kets $\equiv |x\rangle$

Bra vs Ket

define Bra to suit operations

ex: $|v\rangle = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ then $\langle v| = [v_1 \dots v_n]$ to multiplication

ex: $|n\rangle = e^{inx} \rightarrow \langle n| = e^{-inx}$ for real inner product

define an operator $\equiv$ Matrix $\underline{M} \rightarrow \underline{M}|x\rangle = |w\rangle$

are there $|x\rangle$ or $\vec{x}$ & λ s.t. $\underline{M} \cdot \vec{x} = \lambda \vec{x}$

scalar

instead of being unique vector, can $|w\rangle$ be a scaled $|x\rangle$

$\underline{M} \equiv \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ & $|x\rangle = \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\underline{M}\,\vec{x} - \lambda \mathbb{1}\,\vec{x} = 0$

factor out $\vec{x}$

$\rightarrow (\underline{M} - \lambda \mathbb{1}) \cdot \vec{x} = \vec{0}$ $\quad \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

solved by $\vec{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow$ no fun

$[\underline{M} - \lambda \mathbb{1}]^{-1} [\underline{M} - \lambda \mathbb{1}] \cdot \vec{x} = \underbrace{[\underline{M} - \lambda \mathbb{1}]^{-1}}_{\text{what if this } = \infty ?} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

if $(\underline{M} - \lambda \mathbb{1}) \cdot \vec{x} = \vec{0}$, $\vec{x}$ solved by

inverse of a matrix $\sim \dfrac{\text{goes like "cofactor" matrix}}{\text{determinant of matrix}}$

how space changes after a matrix
is it squished or stretched?
by how much?

$\rightarrow$ non trivial sol^{ns} if $\det[\underline{M} - \lambda \mathbb{1}] = 0$

is there a vector s.t. [ ] collapses to lower dimension?

$$\det\left[\begin{pmatrix} a & b \\ b & a \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right] = \det\left[\begin{pmatrix} a-\lambda & b \\ b & a-\lambda \end{pmatrix}\right] = 0$$

$$\det[\quad] = (a-\lambda)\cdot(a-\lambda) - (b)(b) = 0$$

$$\lambda = a \pm b$$

these are the 2 eigenvalues

only 2 particular vectors $\vec{x} = \vec{V}_\pm$ work

$$[\underline{M} - \lambda_+ \mathbb{1}]\,\vec{V}_+ = 0 = \begin{bmatrix} a-(a+b) & b \\ b & a-(a+b) \end{bmatrix} \begin{bmatrix} V_{+1} \\ V_{+2} \end{bmatrix} = \begin{bmatrix} -b & b \\ b & -b \end{bmatrix} \begin{bmatrix} V_{+1} \\ V_{+2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$V_+ \propto \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad V_{+1} \begin{bmatrix} -b \\ b \end{bmatrix} + V_{+2} \begin{bmatrix} b \\ -b \end{bmatrix} = \begin{bmatrix} V_{+1} \cdot -b + V_{+2}\, b \\ V_{+1} \cdot b + V_{+2} \cdot -b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$[\underline{M} - \lambda_- \mathbb{1}]\,\vec{V}_- = 0 = \begin{bmatrix} a-(a-b) & b \\ b & a-(a-b) \end{bmatrix} \begin{bmatrix} V_{-1} \\ V_{-2} \end{bmatrix} = \begin{bmatrix} b & b \\ b & b \end{bmatrix} \begin{bmatrix} V_{-1} \\ V_{-2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_- \propto \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

transposed

$$v_+^T \cdot v_- \propto \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$

→ orthogonal (v_+ & v_-)

always for eigenvalues

$v_{+1} b = v_{+2} b$

$v_{+1} b = v_{+2} b$

$v_{+1} = v_{+2}$