Intro to Linear Vector Spaces PHB 8.1–8.3

coord. vectors $\vec{r} = (x, y, z)$ define an object in vector space

Rules define an algebra $\rightarrow$ linear vector space (LVS)

$$\vec{r}_1 \pm \vec{r}_2 = \vec{r}^3 \quad \text{in same LVS}$$

$$\left.\begin{aligned} &\vec{a} + \vec{b} = \vec{b} + \vec{a} \\ &(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}) \\ &\lambda(\vec{a} + \vec{b}) = \lambda\vec{a} + \lambda\vec{b} \quad \lambda \text{ is scalar} \end{aligned}\right\} \text{linear operations}$$

$$\vec{a} + \vec{0} = \vec{a}$$

$$\vec{a} + (-\vec{a}) = \vec{0}$$

Other Operations: "Multiplication"... Many ways to define $\Rightarrow$ inner products

Dimensions of LVS: Suppose I can find N special vectors, $\vec{v}_i$, s.t. any $\vec{a} = \sum_{i=1}^{N} c_i \vec{v}_i$ $\rightarrow$ dimension N

Basis $\vec{v}_i$, $i = 1 \rightarrow N$, form basis spanning LVS

Linear independence: Suppose you have candidate set $\vec{v}_i$, but you find there are constants α_i s.t.

$$\sum_{i=1}^{N} \vec{v}_i \cdot \alpha_i = \vec{0}$$

$$\longrightarrow \quad \vec{V}_N = \frac{1}{\alpha_N} \cdot \sum_{i=1}^{N-1} \alpha_i \vec{v}_i$$

$\rightarrow$ not indep't, can be made up of others
dimensionality is $N-1$

test for linear independence $\sum_{i=1}^{N} \alpha_i \vec{v}_i \neq \vec{0}$

example 2-D

$\vec{v}_1 = (1, 0)$ $\vec{v}_2 = (0, 1)$ $\vec{v}_3 = (1, 1)$

$1 \cdot \vec{v}_1 + 1 \cdot \vec{v}_2 + (-1)\,\vec{v}_3 = 0$

$\rightarrow N-1$ or $3-1$ or 2 dimensions

if $\vec{v}_1 = (1,0)$ $\vec{v}_2 = (1,1)$ $\rightarrow \sum_{i=1}^{2} a_i \vec{v}_i = (a_1, 0) + (a_2, a_2)$
$= (a_1 + a_2, a_2) = 0$
can only happen if $a_1 = a_2 = 0$, it is a LVS

Different Bases

found that $(1,0)$ & $(0,1)$ work, as does $(1,0)$ & $(1,1)$

example

can write $(2,3) = 2\vec{v}_1 + 3\vec{v}_2$

$(2,3) = \frac{5}{2}(1,1) + -\frac{1}{2}(1,-1)$ if we use $(1,1)$ & $(1,-1)$

$\left(\frac{5}{2} - \frac{1}{2}, \frac{5}{2} + \frac{1}{2}\right)$

new glitch: how to find basis coefficients?

devise an operation (an inner product)

def^{n}: $\langle \vec{a} \,|\, \vec{b} \rangle \equiv$ inner product of $\vec{a}$ & $\vec{b}$ $\equiv$ scalar

must obey rules for LVS:

$\langle \vec{a} \,|\, \vec{b} \rangle = \langle \vec{b} \,|\, \vec{a} \rangle^*$ for complex LVS

$\langle \vec{a} \,|\, \lambda\vec{b} + \mu\vec{c} \rangle = \lambda \langle \vec{a} \,|\, \vec{b} \rangle + \mu \langle \vec{a} \,|\, \vec{c} \rangle$ Note if

$\langle \lambda\vec{a} \,|\, \vec{b} \rangle = \langle \vec{b} \,|\, \lambda\vec{a} \rangle^* = \lambda^* \langle \vec{b} \,|\, \vec{a} \rangle^* = \lambda^* \langle \vec{a} \,|\, \vec{b} \rangle$

for coord. vectors: $\langle \vec{a} \,|\, \vec{b} \rangle = \left\langle \sum_i^N a_i \vec{v}_i \,\middle|\, \sum_j^N b_j \vec{v}_j \right\rangle$

$= \sum_i^N \sum_j^N a_i^* b_j \langle \vec{v}_i \,|\, \vec{v}_j \rangle$

orthogonal basis: we can find $\vec{v}_i$ s.t. $\langle \vec{v}_i | \vec{v}_j \rangle = \delta_{ij}$

normalized: call this basis $\hat{e}_i$ $\quad \langle \hat{e}_i | \hat{e}_i \rangle = 1$ $\to$ normalized

in this basis:
$$\langle \vec{a} | \vec{b} \rangle = \langle \underbrace{\sum_i^N a_i \vec{v}_i}_{\text{"bra"}} | \underbrace{\sum_j^N b_j \vec{v}_j}_{\text{"ket"}} \rangle = \sum_i^N \sum_j^N a_i^* b_j \langle \hat{e}_i | \hat{e}_j \rangle$$
$$= \sum_{i=1}^N a_i^* b_i$$

norm defined: $\|\vec{a}\| = \langle \vec{a} | \vec{a} \rangle = \sum_i a_i^* a_i$ = Real Scalar

8.13.1

How to make an orthonormal basis (Gram-Schmidt)

suppose you have arbitrary $\vec{v}_i$, they do span space

① $\hat{e}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|}$ $\qquad$ start w/ any $\vec{v}_i$

② Modify $\vec{v}_2$ so it has no $\hat{e}_1$ components

component of $\vec{v}_2$ on $\hat{e}_1$

$\hat{e}_2 = \vec{v}_2 - \langle \hat{e}_1 | \vec{v}_2 \rangle \hat{e}_1$, then normalize $\cdot \dfrac{1}{\vec{v}_2 - \langle \hat{e}_1 | v_2 \rangle \hat{e}_1}$

③ Modify $\vec{v}_3$ so that it has no $\hat{e}_1$ or $\hat{e}_2$ component

$\hat{e}_3 = \vec{v}_3 - \langle \hat{e}_1 | \vec{v}_3 \rangle \hat{e}_1 - \langle \hat{e}_2 | \vec{v}_3 \rangle \hat{e}_2$, normalize

example in 3-D $\quad \vec{v}_1 = (5,0,0)$, $\vec{v}_2 = (1,1,1)$, $\vec{v}_3 = (1,0,-1)$

$\hat{e}_1 = \dfrac{(5,0,0)}{5} = (1,0,0)$

before normalization

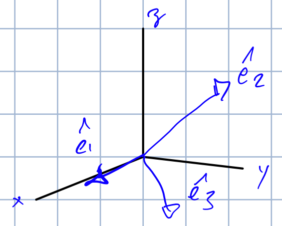
$\sim \hat{e}_2 = \overset{\vec{v}_2}{(1,1,1)} - \langle \overset{\hat{e}_1}{(1,0,0)} | \overset{\vec{v}_2}{(1,1,1)} \rangle \cdot \overset{\hat{e}_1}{(1,0,0)} = (0,1,1)$

$\hat{e}_2 = \frac{1}{\sqrt{2}} (0,1,1)$

$$\tilde{\hat{e}}_3 = (1,0,-1) - \langle (1,0,0) \,|\, (1,0,-1)\rangle \cdot (1,0,0) - \left\langle \frac{(0,1,1)}{\sqrt{2}} \,\middle|\, (1,0,-1)\right\rangle \cdot \left(\frac{(0,1,1)}{\sqrt{2}}\right)$$

$$= (1,0,-1) - (1,0,0) + \frac{(0,1,1)}{2}$$

$$= \left(0, \tfrac{1}{2}, -\tfrac{1}{2}\right) \qquad \hat{e}_3 = \frac{(0,1,-1)}{\sqrt{2}}$$



new orthonormal system