21.3.1 & 18.1.1

Last time saw separation variables, usually w/ power series

$$y(x,t) = R(x)\, S(t)$$

Separate & equate $\qquad \lambda = \frac{R''}{R} = \frac{\dot{S}}{S}$

Solve 2 ODE in terms of unknown λ

Use Boundary Conditions (BC) to get allowed λ_n

Use initial conditions (IC) to pin down all constants

Laplace's Eqn $\qquad \vec{\nabla}^2 \psi(\vec{r}) = 0 \qquad$ for no azimuthal dependence $\mu = \cos\theta$
azimuthal symmetry

in spherical (r, θ, ϕ) polar, azimuthal
no ϕ dependence

$$\vec{\nabla}^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left[(1-\mu^2)\frac{\partial^2}{\partial \mu^2} - 2\mu \frac{\partial}{\partial \mu}\right]$$

Separate $\qquad \psi(r,\mu) = R(r)\, P(\mu)$

PDE

$$\left[\ddot{R} + \frac{2}{r}\dot{R}\right]P + \frac{1}{r^2}\left[(1-\mu^2)P'' \; 2\mu P'\right] = 0$$

$\div RP/r^2$

$$r^2 \frac{R'' + \frac{2}{r}R'}{R} + \frac{(1-\mu^2)P'' - 2\mu P}{P} = 0$$

$$r^2 \frac{R'' + \frac{2}{r}R'}{R} = -\frac{(1-\mu^2)P'' - 2\mu P}{P} = \text{constant} = l(l+1)$$

get Legendre's DE: $(1-\mu^2)P'' - 2\mu P' + l(l+1)P = 0$

lecture 5 $\quad P_l(\mu) = \sum_{n=0}^{\infty} c_n \mu^n \rightsquigarrow \frac{c_{n+2}}{c_n} = \frac{n(n+1) - l(l+1)}{(n+2)(n+1)}$

BC: $P_l(\mu = \pm 1)$ must be finite $\rightarrow l = 0, 1, 2, \ldots$

$\rightarrow$ our sol^{n}'s are the $P_l(\mu)$ Legendre's polynomials $\quad$ 18.1.1

now $R \qquad R'' + \frac{2}{r}R' - \frac{1}{r^2}l(l+1)R = 0$

or $\quad r^2 R'' + 2r R' - l(l+1)R = 0$

$\rightarrow$ try $R_\ell = \sum_{n=0}^{\infty} a_n r^n$

$$0 = \sum_{n=0}^{\infty} \left[n(n-1)\, a_n r^{n-2+2} + 2n\, a_n r^{n-1+1} - \ell(\ell+1)\, a_n r^n \right]$$

$$= \sum_{n=0}^{\infty} \underbrace{\left[n(n-1) + 2n - \ell(\ell+1) \right]}_{=0} a_n r^n$$

only 2 values of n work: $n = \ell$ or $n = -(\ell+1)$

$$R_\ell(r) = A_\ell r^\ell + B_\ell r^{-(\ell+1)}$$

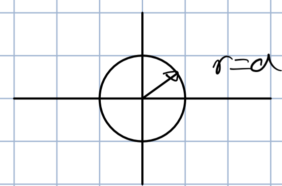
$$P_\ell(r) + R_\ell(r) = 0 = \left(A_\ell r^\ell + B_\ell r^{-(\ell+1)}\right) P_\ell(\mu)$$

$$\psi(r,\mu) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell r^{-(\ell+1)}\right) P_\ell(\mu)$$

if $\psi(r,\mu) \xrightarrow{r\to\infty} 0$, $r^\ell \to \infty \implies A_\ell$ must $= 0$ ($\ell \geq 1$)

if $\psi(r,\mu) \xrightarrow{r=0}$ Finite, $r^{-(\ell+1)} \to \infty \implies B_\ell$ must $= 0$

have to define regions — some spherical boundary



$\rightarrow \psi_{r\leq a}$, $\psi_{r\geq a}$

Example

1) $\nabla^2 \frac{1}{|\vec{r}-\vec{r}_0|} = 0$ for point charge @ $\vec{r}_0$

$\hat{r}\cdot\hat{r}_0 = \cos\theta$ between $\vec{r}$ & $\vec{r}_0$

then for $r < r_0$ $\quad \frac{1}{|\vec{r}-\vec{r}_0|} = \sum_{\ell=0}^{\infty} A_\ell\, r^\ell P_\ell(\hat{r}\cdot\hat{r}_0)$

to find A_ℓ: look @ $\mu = 1$ $\quad \hat{r}\cdot\hat{r}_0 = 1 \rightarrow \theta = 0$

& use $P_\ell(\mu=1) = 1$

& $\frac{1}{|\vec{r}-\vec{r}_0|} = \frac{1}{|r-r_0|}$ w/ $\mu = 1$

$\rightarrow$ match expansion of $\frac{1}{|r-r_0|}$ w/ $\sum_{\ell=0}^{\infty} A_\ell r^\ell$

2) $\varphi(r=a,\mu) = f(\mu)$ known BC

$$f(\mu) = \sum_{\ell=0}^{\infty} A_\ell a^\ell P_\ell(\mu)$$ for sol^n $r \le a$

another property of $P_\ell(\mu)$: orthogonal $\rightarrow$ $\int_{-1}^{1} P_\ell(\mu) P_m(\mu) d\mu = \frac{2}{2m+1} \underbrace{\delta_{\ell m}}_{\text{if } \ell = 1}$

Get A_ℓ

$$\int_{-1}^{1} f(\mu) P_m(\mu) d\mu = \sum_{\ell=0}^{\infty} A_\ell (a^\ell) \cdot \int_{-1}^{1} P_\ell(\mu) P_m(\mu) d\mu$$

$$= A_m \quad a^m \quad \frac{2}{2m+1}$$

$$A_m = \frac{2m+1}{2} a^{-m} \int_{-1}^{1} d\mu \, f(\mu) P_m(\mu)$$

cartesian : $\underbrace{\text{sin, cos,}}_{e^{i\alpha - \beta}}$ fourier

spherical : legendre