Midterm wk 5?
on DEs

Last time claimd $\mathcal{L}(u(x)) = 0$ solved by $u = X(x)\,T(t)$

To begin $y''(x) + \omega^2 y(x) = 0$ oscillator

$$\rightarrow \frac{y''}{y} = -\omega^2 \qquad \sim\text{eigenvalue}$$

find which $-\omega^2$ satisfies boundary conditions

boundary conditions: $y(0) \equiv 0 \equiv y(L)$ $\rightarrow$ taut string

general sol$^{\underline{n}}$: $y(x) = Ae^{i\omega x} + Be^{-i\omega x}$

$$y(0) = 0 = A + B \rightarrow B = -A \rightarrow y = A(e^{i\omega x} - e^{-i\omega x})$$

$$\rightarrow y = \underset{\text{old } A \cdot 2}{A} \sin(\omega x)$$

$$y(L) = 0 = A\sin(\omega L) \rightarrow \sin(\omega L) = 0$$

$$\rightarrow \omega L = n\pi \quad , \; n = \text{integer}$$

if we only consider $\omega > 0$, $\omega = n\frac{\pi}{L}$, $n = 0, 1, 2, \ldots$

$\rightarrow$ these ω are eigenvalues of DE w/ boundary conditions

Back to PDE $\quad U_{xx}(x,t) = \frac{1}{c^2} U_{tt}(x,t)$ wave eq$^{\underline{n}}$

try $U = X(x)\,T(t)$

$$X''\,T = \frac{1}{c^2} X \ddot{T}$$

rewrite as $\quad \downarrow \frac{1}{XT}$

$$\underbrace{\frac{X''}{X}}_{f^{\underline{n}} \text{ of } (x)} = \underbrace{\frac{1}{c^2}\frac{\ddot{T}}{T}}_{f^{\underline{n}} \text{ of } (t)} = C \underset{\text{looking ahead}}{\equiv} -k^2$$

must hold for any x, t $\rightarrow$ must be equal to some constant get eq^{n}'s $\quad \ddot{X} = -k^2 X$

$$X = Ae^{ikx} + Be^{-ikx}$$

boundary conditions: $X(0) = 0 \rightarrow 0 = A + B \rightarrow B = -A \rightarrow X(x) = A\sin(kx)$

$$X(L) = 0 \rightarrow kL = n\pi \rightarrow k = n\frac{\pi}{L} = k_n$$

next

$$\ddot{T} = -k_n^2 c^2 T$$

$$T(t) = C_n e^{ik_n ct} + D_n e^{-ik_n ct}$$

let $k_n c = \omega_n$

$$T(t) = C_n \cos(\omega_n t) + D_n \sin(\omega_n t)$$

$$u(x,t) = \sum_{n=0}^{\infty} \sin\left(\frac{n\pi}{5}x\right)\left[C_n \cos\left(n\frac{\pi c}{L}t\right) + D_n \sin\left(n\frac{\pi c}{L}t\right)\right]$$

need more info to get A_n + D_n $\rightarrow$ initial conditions

a possible initial condition could be $u(x, t=0) = f(x) = A\sin\left(\frac{5\pi}{2}x\right)$ made up

to get C_n D_n use orthogonality of sin + cos

@ $t=0$:

$$f(x) = A\sin\left(\frac{5\pi x}{2}\right) = \sum_{n=0}^{\infty} \sin\left(\frac{n\pi x}{2}\right)\left[\underbrace{C_n \cos(0)}_{\to C_n} + \underbrace{D_n \sin(0)}_{\to 0}\right]$$

pick any m

$$\int_0^L A\sin\left(\frac{5\pi x}{2}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \sum_{n=0}^{\infty}\int_0^2 C_n \sin\left(\frac{n\pi x}{2}\right)\sin\left(\frac{m\pi x}{L}\right)dx$$

$u = \frac{\pi x}{L}$

$du = \frac{\pi}{L}dx$

$$= \sum_{n=0}^{\infty} C_n \frac{2}{\pi}\underbrace{\int_0^2 \sin(nu)\sin(mu)\,du}_{\frac{L}{\pi}\frac{\pi}{2}\delta_{mn}}$$

$\rightarrow$ 0 if $m \neq n$, 1 if $m = n$

$$A\frac{L}{2}\delta_{m5} = C_m \frac{L}{2}$$

only C_5 is nonzero $\qquad C_m = A$

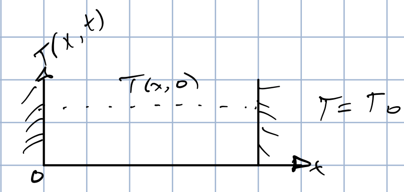
$$u(x,t) = A\sin\left(\frac{5\pi x}{2}\right)\cos\left(\frac{5\pi ct}{L}\right)$$

some boundary conditions fix

some initial conditions, use orthogonality

will find all $D_n = 0$ by multiplying by $\cos\left(\frac{m\pi}{2}x\right)$

Thermal Diffusion



$T(x,t)$ = temp.

$$\frac{\partial^2 T}{\partial x^2} = \chi \frac{\partial T}{\partial t} \rightarrow T(x,t) = R(x)\, S(t) \rightarrow R'' S = \chi R S'$$

$$\frac{R''}{R} = \chi \frac{S'}{S} = -\lambda^2$$

$$S' = -\frac{\lambda^2}{\chi} S \rightarrow S(t) = S_0 e^{-\frac{\lambda^2}{\chi} t}$$

$$R'' = -\lambda^2 R \rightarrow R(x) = A\cos(\lambda x) + B\sin(\lambda x)$$

Boundary conditions: $T(0,t) = T_0 = 0 \rightarrow R(0) = 0 \rightarrow A = 0$

$T(L,t) = T_0 = 0 \rightarrow \lambda_n = \frac{n\pi}{L}$

$$T(x,t) = \sum_{n=0}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{(n\pi/L)^2}{\chi} t}$$

initial condition: $T(x, t=0) = T_0 = \sum_n B_n \sin\left(\frac{n\pi x}{2}\right) \cdot 1 \leftarrow e^0$

$$T_0 \int_0^L \sin\left(\frac{m\pi x}{2}\right) dx = \sum_{n=0}^{\infty} B_n \underbrace{\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{2}\right) dx}_{\frac{L}{2}\delta_{mn}}$$

$$T_0 \frac{2}{m\pi}(1-\cos(m\pi))$$

$B_m = \frac{4T_0}{m\pi}$ for m odd, $B_m = 0$ for m even

$$T(x,t) = \sum_{n=0}^{\infty} \frac{4T_0}{\pi(2n+1)} \sin\left(\frac{(2n+1)\pi x}{2}\right) e^{-\frac{(2n+1)^2\pi^2}{\chi L^2} t}$$

next lecture: polar coords