Partial Differential Equations

Review what we know about ODEs no Wronskian or integrating factor

① First order separable $y'(x) = \frac{f(x)}{g(y)}$

$$\int g(y)\,dy = \int f(x)\,dx$$

② Second order constant coefficients, RHS = 0

Factorization: $y''(x) + A(x)\,y'(x) + B(x)\,y(x) = \left(\frac{d}{dx} - a\right)\left(\frac{d}{dx} - b\right) y(x)$

$$\rightarrow \quad y(x) = \left[C_1 e^{ax} + C_2 e^{bx}\right] \cdot \underbrace{(ax^n + bx^{n-1} + \dots)}_{\text{sometimes}}$$

③ RHS ≠ 0

$$y'' + Ay' + By = f(x) \quad e^{ax} \quad \text{or} \quad \mathrm{Re}\left[e^{i\omega x}\right]$$

$$y_g(x) = \underbrace{y_c(x)}_{\text{when RHS}=0} + y_p(x)$$

→ has 2 integrating factors

④ Non-constant coefficients → try expanding by bases

try $y(x) = x^s \sum_{n=0}^{\infty} c_n x^n$

organize DE Σ in powers of x^n

recursion relation: $c_{n+2} = \dots \cdot c_n$

RHB chp. 20 PDEs intro, go to ch 21

Useful PDEs:

E&M: $\nabla^2 \varphi(x,y,z) = \rho(x,y,z)$ NB: $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

Waves: $\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right] \vec{E}(\vec{r},t) = 0$

Heat: $\left[\nabla^2 - \alpha \frac{\partial}{\partial t}\right] u(\vec{r},t) = 0$

2HB intro. : try to be clever

consider wave eqⁿ in 1 spacial dimension

$$\frac{\partial^2}{\partial x^2} U(x,t) = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} U(x,t)$$

what if $U(x,t) = U(p(x,t))$ such as $p(x,t) = ax + bt$

$$\frac{\partial U}{\partial x} = U_x = \underbrace{U'}_{\frac{dU}{dp}} \underbrace{P_x}_{\frac{\partial P}{\partial x}}$$

$$U_{xx} = (U')_x P_x + U'(P_x)_x = U'' P_x^2 + U' P_{xx}$$

$$U_t = U' P_t$$

$$U_{tt} = (U')_t P_t + U'(P_t)_t = U'' P_t^2 + U' P_{tt}$$

$$U'' P_x^2 + U P_{xx} = \frac{1}{c^2}\left[U'' P_t^2 + U' P_{tt} \right]$$

$$U''\left[P_x^2 - \frac{1}{c^2} P_t^2 \right] + U' \underbrace{\left[P_{xx} - \frac{1}{c^2} P_{tt} \right]}_{=0 \text{ if } P_{xx} = P_{tt} = 0} = 0$$

try $P(x,t) = ax + bt$ $\quad P_x = a \quad P_t = b \quad P_{xx} = 0 = P_{tt}$

$$U''\left[a^2 - \frac{b^2}{c^2} \right] = 0$$

$a^2 = \frac{b^2}{c^2}$ $\quad b^2 = a^2 c^2$ works for $b = \pm ca \rightarrow P_t = \pm c P_x$

$\longrightarrow$ any fⁿ $U(x \pm ct)$ solves the DE

check: $U(x,t) = f(x+ct) + g(x-ct)$

$U_x = f' \cdot 1 + g' \cdot 1 \qquad U_{xx} = f'' \cdot 1 + g''$

$U_t = f' \cdot c - g' c \qquad U_{tt} = f'' c^2 - g''(-c^2)$

$$\text{DE} = U_{xx} - \frac{1}{c^2} U_{tt} = (f'' + g'') - \frac{1}{c^2}(f'' + g'') = 0$$

RHB ch 21

"Separation of variables sol^{n}"

$u(x,t) = X(x) \cdot T(t)$

why can we trust this?
b/c we will have $X(x)$ which form expansion basis

Review stuff

RHB 4.6

Taylor expansion   approximation near point

$$f(x) = f(x_0) + (x-x_0) \cdot f_x\Big|_{x_0} + \frac{1}{2!}(x-x_0)^2 f_{xx}\Big|_{x_0} + \dots$$

$$f(x,y) = f(x_0,y_0) + \left\{(x-x_0)\cdot f_x\Big|_{x_0,y_0} + (y-y_0)\cdot f_y\Big|_{x_0,y_0}\right\}$$
$$+ \frac{1}{2!}\left\{(x-x_0)^2 f_{xx}\Big|_{x_0,y_0} + (y-y_0)^2 f_{yy}\Big|_{x_0,y_0} + 2(x-x_0)(y-y_0) f_{xy}\Big|_{x_0,y_0}\right\}$$
$$+ \dots$$

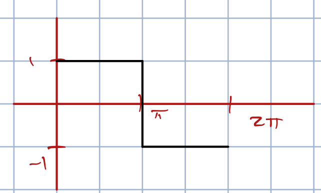
$$e^{x-0} = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$$ example

RHB 12

Fourier series for f^n's in defined over $x = [0, 2\pi]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(nx) + b_n \sin(nx)\right]$$

squarewave



$$f(x) = \frac{4}{\pi}\left[\sin(x) + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{3} + \dots\right]$$

coefficients found using "orthogonality" of sin, cos

$\int_0^{2\pi} \sin(nx)\cos(mx)\,dx = 0$ — For n,m integers

$\int_0^{2\pi} \sin(nx)\sin(mx)\,dx = \pi$ — if $n=m$, 0 otherwise

$\int_0^{2\pi} \cos(nx)\cos(mx)\,dx = \pi$ — if $n=m>0$, 0 if $n\neq m$, 2π if $m=n=0$

$$\longrightarrow \int_0^{2\pi} f(x)\sin(mx)\,dx = \sum_{n>0} b_n \int_0^{2\pi} \sin(mx)\sin(nx)\,dx = \pi b_m$$

$$\longrightarrow \int_0^{2\pi} f(x)\cos(mx)\,dx = \sum_{n>0} \quad = \pi a_m \quad m>0,\ 2\pi a_m \quad m=0$$