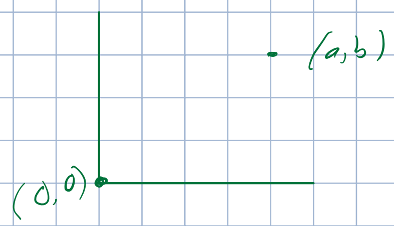
Last time: $\frac{\partial F(y,y',x)}{\partial y} = \frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right]$ or $\frac{\partial F}{\partial x} = \frac{d}{dx}\left[F - y'\frac{\partial F}{\partial y'}\right]$

no explicit x dependence

if F has no explicity: $\frac{\partial F}{\partial y} = \text{Const.}$

example: distance in 2D



Functional to minimize: $J = \int_{0,0}^{a,b} ds = \int_0^a \sqrt{1+y'^2}\,dx$

$(x,y)=$ $\quad$ $x=$

$\rightarrow F = \sqrt{1+y'^2}$

$F_{y'} = \text{const.} = \frac{y'}{\sqrt{1+y'^2}} \rightarrow c^2(1+y'^2) = y'^2$

$\rightarrow c^2 + c^2 y'^2 = y'^2 \rightarrow y'^2(1-c^2) = c^2$

$y' = \pm\sqrt{\frac{c^2}{1-c^2}} = \text{const.}$

$y = c_1 x + c_2$

How to do in 3-D?

$y' = \frac{d}{dx}[y(x)]$ $\quad z' = \frac{d}{dx}(z(x))$

$ds = \sqrt{dx^2 + dy^2 + dz^2} = \sqrt{1 + y'^2 + z'^2}\,dx$

try same type of derivation as last time

sol^n's

$y(x) = \bar{y}(x) + \epsilon\, n_1(x)$ $\quad$ w/ $n_1(a) = n_1(b) = 0$ endpoints

$z(x) = \bar{z}(x) + \epsilon\, n_2(x)$ $\quad$ w/ $n_2(x) = 0$ @ endpoints

$$\left.\frac{dJ}{d\epsilon}\right|_{\epsilon=0} = \int dx \cdot \frac{d}{d\epsilon}[F] = \int dx \left[\frac{\partial F}{\partial y}\cdot\underbrace{\frac{dy}{d\epsilon}}_{n_1} + \frac{\partial F}{\partial y'}\cdot\underbrace{\frac{dy'}{d\epsilon}}_{n_1'} + \frac{\partial F}{\partial z}\cdot\underbrace{\frac{dz}{d\epsilon}}_{n_2} + \frac{\partial F}{\partial z'}\cdot\underbrace{\frac{dz'}{d\epsilon}}_{n_2'}\right]$$

integrate by parts for both n_1' & n_2' $\Rightarrow$

$$0 = \int dx \left\{ n_1(x)\cdot\underbrace{\left(F_y - \frac{d}{dx}[F_{y'}]\right)}_{\rightarrow 0} + n_2(x)\underbrace{\left(F_z - \frac{d}{dx}[F_{z'}]\right)}_{\rightarrow 0}\right\}$$

$\rightarrow = 0$ for any n_1 or n_2

have E-L for each

E-L eqⁿ for each dependent variable:

$$\frac{\partial F(y_1, y_1', \dots, y_n, y_n')}{\partial y_i} = \frac{d}{dx}\left[\frac{\partial F}{\partial y_i'}\right]$$

For 3-D distance: $F = \sqrt{1+y'^2+z'^2}$

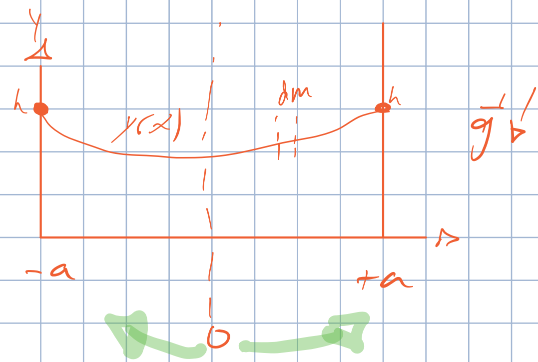
no explicit x

$$F_{y'} = C_1 = \frac{y'}{\sqrt{1+y'^2+z'^2}} \qquad F_{z'} = C_2 = \frac{z'}{\sqrt{1+y'^2+z'^2}}$$

$$\hat{y}(x) = Ax+B \qquad \tilde{z}(x) = Cx+D$$

Constrained Problems

Shape of hanging rope: ("catenary")



constrain: length L

$ds = \sqrt{1+y'^2}\,dx$

constraint: $\int_{-a}^{a} ds = L$

physics: minimize PE: $dU = dm\, g\, y(x)$, $dm \to \rho \cdot ds$

$$U = \rho \cdot g \cdot \int_{-a}^{a} \underbrace{dx\sqrt{1+y'^2}\cdot y}_{\to F(y,y';x)}$$

define $G \equiv \sqrt{1+y'^2} - \frac{L}{2a}$ (constant) s.t. $\int_{-a}^{a} G\,dx = 0$

→ adjust functional

$$J \to \int \underbrace{(F + \lambda G)}_{\tilde{F}\text{ - original } F \text{ w/ constraint}} dx$$

λ: lagrange multiplier

derivative

$y\sqrt{1+y'^2} + \lambda\left(\sqrt{1+y'^2} - \frac{L}{2a}\right)$

$y\sqrt{1+y'^2} + \lambda\sqrt{1+y'^2} - \lambda\frac{L}{2a}$

$\sqrt{1+y'^2}\,(y+\lambda) \underbrace{- \lambda\frac{L}{2a}}_{\text{absorbed by } C}$

$$\tilde{F} = \sqrt{1+y'^2}\,(y+\lambda)$$

no explicit x: $\tilde{F}_x = 0 \to \tilde{F} - y'\cdot\frac{\partial\tilde{F}}{\partial y'} = \text{const.}$

$$C = \sqrt{1+y'^2}\cdot(y+\lambda) - y'(y+\lambda)\frac{y'}{\sqrt{1+y'^2}} = \frac{y+\lambda}{\sqrt{1+y'^2}}\left(1+y'^2-y'^2\right)$$

$$c^2 y'^2 = (y+\lambda)^2 - c^2 \longrightarrow y' = \frac{\sqrt{(y+\lambda)^2 - c^2}}{c^2}$$ seperable

$u = y + \lambda$

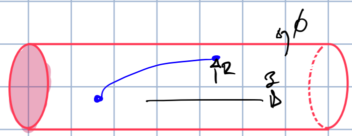
$$y + \lambda = C \cosh\left(\frac{x - x_0}{c}\right)$$

get constants from $\pm x$ symmetry & L

Another type of example: geodesics

consider

constrained on cylinder surface



$$ds = \sqrt{dr^2 + dz^2 + (r\,d\phi)^2}$$

Choose z to be indpt. $\frac{d}{dz}$

dist. between 2 points

$$F = \sqrt{1 + r'^2 + r^2\phi'^2}\, dz$$

but, need to be on surface

constraint: $r = a$ fixed cylinder radius: $G = r - a = 0$

a const by $\frac{d}{dz}$

$$\tilde{F} = \sqrt{1 + r'^2 + r^2\phi'^2} + \lambda \cdot r$$

E-L eq^n: ① $F_r = \frac{d}{dz}\left[\tilde{F}_{r'}\right]$ $r = a \rightarrow r' = 0$

$$\lambda + \frac{r\phi'^2}{\sqrt{1 + r'^2 + (r\phi)^2}} = 0$$

$$-\lambda = \frac{a\,\phi'^2}{\sqrt{1 + a^2\phi'^2}}$$

② $F_\phi - \frac{d}{dz}\left[\tilde{F}_{\phi'}\right]$ no explicit ϕ

$$\tilde{F}_{\phi'} = C = \frac{a^2\phi'}{\sqrt{1 + a^2\phi'^2}} \longrightarrow \phi' = \pm \frac{c}{a^2(a^2 - c^2)} = \text{Constant}$$

along z, $d\phi\, dz$

Note: could have been tricky & unrolled cylinder

then roll again