Euler-Lagrange $\qquad y = y(x)$

Previously: $\quad J[y] = \int_a^b F(y, y'; x)\, dx \quad$ to be extreme for specific y

There is a way to do this for $F(y, y'; x)$ & $y(a)$ & $y(b)$ are specified

Let $\tilde{y}(x)$ be the $y(x)$ that makes $J[y]$ stationary

some arbitrary $y(x)$ can be written as $y(x) = \tilde{y}(x) + n(x)$

$\longrightarrow$ note: $n(a) = n(b) = 0$

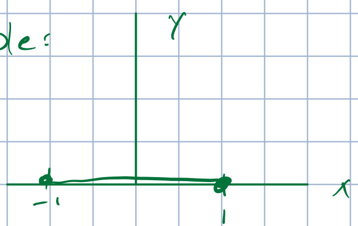
also, no loss of generality if $y(x) = \tilde{y}(x) + \epsilon n(x)$

$\longrightarrow \quad y \xrightarrow{\epsilon \to 0} \tilde{y}$

we consider $J[y] \to J[\epsilon]$ is extreme @ $\epsilon = 0$

$\longrightarrow \quad \frac{dJ}{d\epsilon} = 0 \quad @ \quad \epsilon = 0$

example:



distance $[-1,0] \to [1,0]$ min.

we know $\tilde{y}(x) = 0$

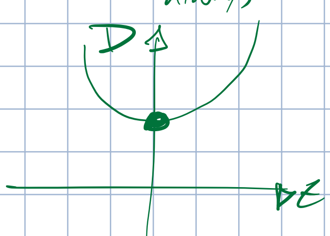
consider $n = x^2 - 1 \qquad$ NB: $n(-1) = n(1) = 0$

$$y(x) = \tilde{y}(x) + \epsilon(x^2 - 1) = 0 + \epsilon(x^2 - 1)$$

$$\text{dist: } \int_{-1,0}^{1,0} ds = \int_{-1}^{1} \sqrt{1 + y'^2}\, dx = \int_{-1}^{1} \sqrt{1 + (\epsilon 2x)^2}\, dx$$

Taylor in ϵ b/c ϵ is small always

$$\cong \int_{-1}^{1} (1 + 2\epsilon^2 x^2)\, dx = 2 + \frac{4\epsilon^2}{3} = D$$



stationary point is $\frac{dD}{d\epsilon} = 0 = \frac{8}{3}\epsilon \longrightarrow \epsilon = 0$

$$y' = y'(x) = \frac{dy}{dx}$$

$$J[\epsilon] = \int_a^b F(y, y'; x)\, dx \quad w/ \quad y(x) = \tilde{y}(x) + \epsilon n(x)$$
$$y' = \tilde{y}' + \epsilon n'$$

$$\frac{d}{d\epsilon}[y] = \underbrace{\frac{d}{d\epsilon}[\tilde{y}]}_{\to 0} + \underbrace{\frac{d}{d\epsilon}[\epsilon n(x)]}_{\to n(x)}$$

similar for $y' \to \frac{dy'}{d\epsilon} = n'(x)$

stationary @ $\left.\frac{dJ}{d\epsilon}\right|_{\epsilon=0} = 0$

Chainrule $\frac{dF}{d\epsilon} = \frac{\partial F}{\partial y} \cdot \frac{dy}{d\epsilon} + \frac{\partial F}{\partial y'} \cdot \frac{dy'}{d\epsilon}$

$$\longrightarrow 0 = \int_a^b dx \frac{dF}{d\epsilon} = \int_a^b dx \left[F_y \cdot \underbrace{\frac{dy}{d\epsilon}}_{n(x)} + F_{y'} \cdot \underbrace{\frac{dy'}{d\epsilon}}_{n'(x)} \right]_{\epsilon=0}$$

$$F_y = \frac{\partial F}{\partial y}$$

integrate 2nd term

$$\int dx\, n' F_{y'} = \int F_{y'}\, dn = \underbrace{F_{y'} n(x)\Big|_a^b}_{0} - \int dF_{y'}\, n$$

$$= \int n \frac{d}{dx} F_{y'}\, dx$$

$$0 = \int_a^b dx \left[F_y - \frac{d}{dx}[F_{y'}] \right]_{\epsilon=0} n(x)$$

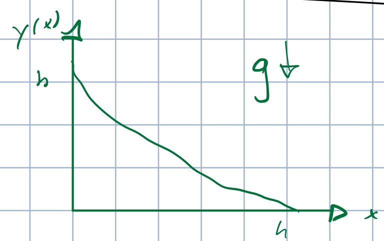
to be true for all $n(x)$,

$$F_y - \frac{d}{dx}[F_{y'}] = 0$$

Euler-Lagrange Eqn: $\frac{\partial F}{\partial y} = \frac{d}{dx} \frac{\partial F}{\partial y'}$

Example: Brachistochrone

$$J[y] = \int \frac{ds}{\sqrt{y}}$$



$$ds = \sqrt{dx^2 + dy^2} = \sqrt{(x')^2 + 1}\, dy$$

$$J = \int_0^h \frac{\sqrt{1 + y'^2}}{\sqrt{y}}$$

$F(y, y', x)$ or $\int_x^0 \sqrt{1 + (x')^2} \frac{dy}{\sqrt{y}}$

$F((x)', x; y)$

no explicit x

$$\overbrace{F_x = \frac{d}{dy} F_{x'} \quad = 0}^{\text{no explicit } x}$$

$$\longrightarrow \frac{d}{dy}\left[\frac{1}{\sqrt{y}} \cdot \frac{1}{2} \cdot \frac{2x'}{\sqrt{1 + x'^2}} \right] = 0$$

so $$\frac{1}{\sqrt{y}}\frac{x'}{\sqrt{1+x'^2}} = C$$

$$\frac{x'^2}{1+x'^2} = C^2 y$$

$$\to (x')^2(1-c^2y) = C^2 y$$

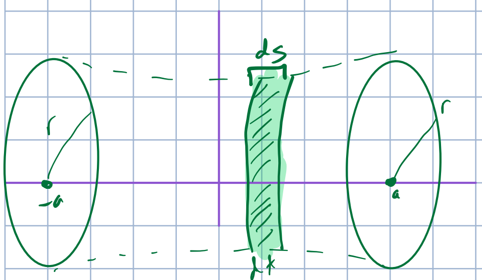
$$\to x' = \sqrt{\frac{c^2y}{1-c^2y}}$$

$$\int_{x(h)}^{x(y)} dx = \int_h^y \sqrt{\frac{c^2y}{1-c^2y}}\,dy \qquad u = c^2y$$

$$x(y) - \underbrace{x(h)}_{\to\,=0} = \frac{1}{c^2}\int_0^{c^2y}\sqrt{\frac{u}{1-u}}\,du$$

$$= h\cos^{-1}\left(\frac{h-y}{h}\right) - \sqrt{2hy-y^2}$$

Another problem: surface of revolution. which $y(x)$ minimizes S.A.?



$$dA = 2\pi y\, ds = 2\pi y\sqrt{1+y'^2}\,dx$$

$$J[y] = \underbrace{2\pi}_{\text{ignore}} \int_{x=-a}^{a} \underbrace{y\sqrt{1+y'^2}}_{F(y,\,y';\,x)}\,dx$$

E-L: $$\frac{\partial F}{\partial y} = \sqrt{1+y'^2}$$

$$\frac{\partial F}{\partial y'} = y\frac{y'}{\sqrt{1+y'^2}}$$

$$\sqrt{1+y'^2} = \frac{d}{dx}\left[y\frac{y'}{\sqrt{1+y'^2}}\right]$$

$$= \frac{(y'^2+y\cdot y'')\sqrt{1+y'^2} - y\cdot y'^2}{\sqrt{1+y'^2}^{\,3}}$$

Another form of E-L b/c ↑ messy

"2nd Form"

$$F_y = \frac{d}{dx}F_{y'}$$

$$\frac{d}{dx}(F_y) = F_y\cdot y' + F_{y'}\,y'' + F_x\cdot x'$$

notice: $\frac{d}{dx}[F_{y'} \cdot y'] = y'' F_{y'} + y' \cdot \frac{d}{dx}[F_{y'}]$

$$\frac{d}{dx}[F - F_{y'} \cdot y'] = (F_y y' + F_{y'} y'' + F_x) - (y'' F_{y'} + y' \frac{d}{dx} F_{y'})$$

$$= F_x + y' \underbrace{(F_y - \frac{d}{dx} F_{y'})}_{E\text{-}L \; = 0}$$

$$= F_x$$

$$F_x = \frac{d}{dx}[F - F_{y'} \cdot y']$$

if no explicit x, $F_x = 0$

Surface problem: $F - y' F_{y'} = C = y\sqrt{1 + y'^2} - y'\left(y \frac{y'}{\sqrt{1+y'^2}}\right) = \frac{y}{1 + y'^2}$

$$\longrightarrow \quad y' = \frac{1}{c} \sqrt{y^2 - c^2}$$

$$y = C \cosh\left(\frac{x}{C} - \frac{x_0}{C}\right)$$