Constraint & function for extremezation

Where is max/min of $f(x,y) = (x-a)^2 + by^2 - c$

$x$ & $y$ are indpt, so find $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$

$\rightarrow 2(x-a) = 0 \qquad 2by = 0 \rightarrow (a, 0)$ is an extrema

Same as : $df = 0 = \underbrace{\frac{\partial f}{\partial x}}_{f_x} dx + \underbrace{\frac{\partial f}{\partial y}}_{f_y} dy$

But, what if $x$ & $y$ are constrained by $g(x,y) = 0$ ?

$x$ & $y$ no longer indpt., are related by $dg = 0 = g_x dx + g_y dy$

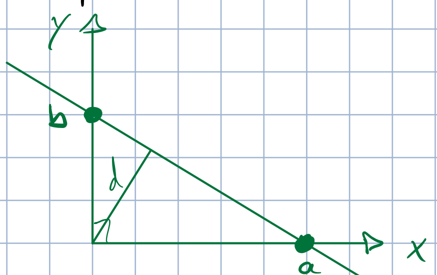
$\rightarrow dy = -\frac{g_x}{g_y} dx$

so $df = 0$ becomes $0 = f_x dx + f_y dy$

$\rightarrow \quad = \left[ f_x - \frac{g_x}{g_y} f_y \right] dx$

$\rightarrow \quad f_x g_y = f_y g_x$

Example : Shortest distance from origin to line $y = -\frac{b}{a}x + b$




$f = d^2 = x^2 + y^2$    $f^n$, distance for any point

$g = bx + ay - ab = 0$    $g$, constraint

$f_x g_y = 2x \cdot a$

$f_y g_x = 2y \cdot b$

$\rightarrow x \cdot a = y \cdot b$

$y_{ex} = \frac{a}{b} x_{ex} \rightarrow x_{ex} = \frac{b}{a} y_{ex}$    ex → extreme

plug into constraint $g$ $\rightarrow$ $y_{ex} = -\frac{b}{a} \cdot \left( \frac{b}{a} y_{ex} \right) + b$

$y_{ex} = \frac{a^2 b}{a^2 + b^2} \qquad x_{ex} = \frac{ab^2}{a^2 + b^2}$

$f = d^2 = x_{ex}^2 + y_{ex}^2 = \frac{a^2b^2}{a^2+b^2}$

Method can get hard fast

Lagrange Multiplier trick

$df + \lambda dg = 0$ $\lambda$ is some constant

$\rightarrow df + \lambda dg = 0 = (f_x + \lambda g_x)\,dx + (f_y + \lambda g_y)\,dy = 0$ but $dx$ & $dy$ are not indpt.

I can choose $\lambda$ s.t. $(f_y + \lambda g_y) = 0 \rightarrow (f_x + \lambda g_x) = 0$

Apply to last example:
$$\boxed{\begin{aligned} f_y + \lambda g_y &= 2y + \lambda a = 0 \\ f_x + \lambda g_x &= 2x + \lambda b = 0 \end{aligned}}$$

$$\begin{aligned} &2by + \lambda ab = 0 \\ -\,(&2ax + \lambda ab = 0) \\ \hline &2by - 2ax = 0 \end{aligned} \rightarrow y = \frac{a}{b}x \quad \text{like before}$$

usually, $g(x,y)$ constraint = 3rd eqn $\rightarrow$ get $\lambda$

generally, $\lambda$ is meaningless

in mechanics, $\lambda$ = Force of Constraint

for our example: $\lambda = \frac{-2ab}{a^2+b^2} = \frac{-2}{ab} d^2$ ... no meaning

Example in QM

Lowest energy state of a particle of mass $m$ "in a box"

$$E_0 = \frac{\hbar^2}{8m}\left[\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right]$$ for rectangular box $a \cdot b \cdot c$

What shape box $(a,b,c)$ gives lowest $E_0$ for constraint $abc = V_0$

$V_0 = xyz$ want $g = 0$

$$f = \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} \qquad g = xyz - V_0$$

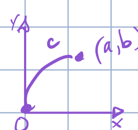
3 Lagrange Eq$^{n}$'s:

$$\begin{aligned}
f_x + \lambda g_x &= -2x^{-3} + \lambda yz = 0 \xrightarrow{\cdot x} \\
f_y + \lambda g_y &= -2y^{-3} + \lambda xz = 0 \xrightarrow{\cdot y} \\
f_z + \lambda g_z &= -2z^{-3} + \lambda xy = 0 \xrightarrow{\cdot z}
\end{aligned}
\quad
\begin{bmatrix}
-2x^{-2} + \lambda xyz = 0 \\
-2y^{-2} + \lambda xyz = 0 \\
-2z^{-2} + \lambda xyz = 0
\end{bmatrix}$$

$\longrightarrow$ $x = y = z$ cube is extreme shape

What is $\lambda$? $\lambda = \frac{2x^{-2}}{V_0} = \frac{2(V_0^{1/3})^{-2}}{V_0} = 2V_0^{-5/3}$ useless

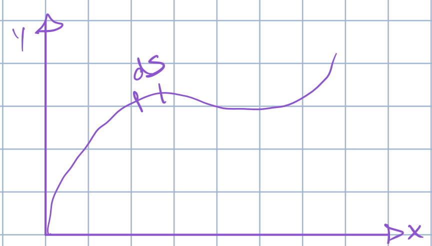
Now ask a different q$^{n}$:

For 

what shape of curve $c$ gives min. dist.?

Set up problem:

line integral





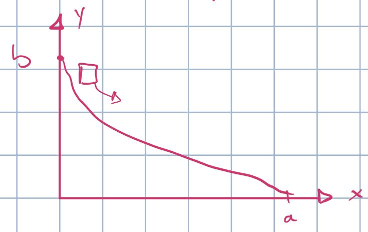
$$ds = \sqrt{dx^2 + dy^2}$$

distance: $d = \int_{(0,0)}^{(a,b)} ds$ is to be minimized by shape $y(x)$

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dx = \sqrt{1 + y'(x)^2}\, dx$$

So : $d = \int_{(0,0)}^{(a,b)} \sqrt{1+(y'(x))^2}\,dx$ is to be minimized

Another example: Brachistochrone




What shape gives shortest dist.?

For friction-free motion under gravity

$$\Delta t = \int dt = \int \frac{ds}{ds/dt} = \int \frac{ds}{v(x,y)}$$

get $v(x,y)$ from physics $\rightarrow$ $v(x,y) = \sqrt{2g}\,\sqrt{b-y}$

$$\Delta t \propto \int_{x=0}^{a} \frac{\sqrt{1+y'(x)^2}\,dx}{\sqrt{b-y}}$$

$\downarrow$

$f^n(y(x), y'(x); x)$ ← indpt. variable