Midterm: :)
another midterm & Final :(

Hermitian operators

last time: unitary operator $\Omega^*_{ij} = \langle i | \Omega^* | j \rangle = \Omega^{-1}_{ji}$

1 special operator →

Hermitian operator: $\mathcal{L}^+ = [\mathcal{L}^T]^* = \mathcal{L}$ (transposed & conjugate) different operator

nice properties

suppose $|v\rangle$ & $|w\rangle$ are eigenstates

→ $\mathcal{L}|v\rangle = \lambda_v |v\rangle$ $\qquad$ $\mathcal{L}|w\rangle = \lambda_w |w\rangle$

$M \vec{v} = \lambda \vec{v}$

$\langle v | \mathcal{L} | w \rangle = \langle v | \lambda_w | w \rangle = \lambda_w \langle v | w \rangle$

$\langle w | \mathcal{L}^+ | v \rangle^*$

hermitian $\mathcal{L} = \mathcal{L}^+$ $\quad = \langle w | \mathcal{L} | v \rangle^* = \langle w | \lambda_v | v \rangle^* = \lambda_v^* \langle v | w \rangle$

→ $\lambda_w \langle v|w\rangle = \lambda_v^* \langle v | w \rangle$ → $(\lambda_w - \lambda_v^*) \langle v | w \rangle = 0$

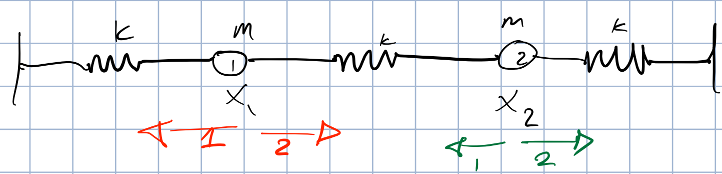
Let $w \to v$: $\quad (\lambda_v - \lambda_v^*) \langle v | v \rangle = 0 \to \lambda_v = \lambda_v^*$ → Real!

eigenvalues of Hermitian operators $\in \mathbb{R}$

if $v \neq w$, $\lambda_v \neq \lambda_w$: b/c $\langle v | w \rangle = 0$ → orthogonal eigenvectors!

Hermitian & Unitary operators generate orthogonal basis!

Example: coupled oscillator



$F_1 = m\ddot{x}_1 = -kx_1 - k(x_2 - x_1) = -2kx_1 + kx_2$

$F_2 = m\ddot{x}_2 = -k(x_2 - x_1) - kx_2 = kx_1 - 2kx_2$

$$\frac{d^2}{dt^2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{k}{m}\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

A, hermitian: $A^T = A$

eigenvalues: $\det\begin{bmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{bmatrix} = 0 = (\lambda+2)^2 - 1 = \lambda^2 + 4\lambda + 3$

$$\lambda = -2 \pm 1$$

$\lambda_1 \equiv -1 : \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \vec{v}_+ \rightarrow \vec{v}_+ = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}}$

eigenvectors

$\lambda_2 \equiv -3 : \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{v}_- \rightarrow \vec{v}_- = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \frac{1}{\sqrt{2}}$

$\vec{v}_+ \cdot \vec{v}_- = \frac{1 \cdot 1 + 1 \cdot -1}{2} = 0$ ✓ nice b/c it's a hermitian matrix

orthogonal, expected b/c hermitian

new primed basis

$$\begin{bmatrix} \ddot{x}_1' \\ \ddot{x}_2' \end{bmatrix} = \omega_0^2 \underbrace{\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}}_{D} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} \rightarrow \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} C_1 e^{i\omega_0 t} \\ C_2 e^{i\sqrt{3}\omega_0 t} \end{bmatrix}$$

let's go back to old basis. use matrix of eigenvectors

original frame

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underline{V} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \left[\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} C_1 e^{i\omega_0 t} \\ C_2 e^{i\sqrt{3}\omega_0 t} \end{bmatrix} \right]$$

$C_1 = \frac{C_1}{\sqrt{2}}$

$C_2 = \frac{C_2}{\sqrt{2}}$

$$\rightarrow = \begin{bmatrix} C_1 e^{i\omega_0 t} + C_2 e^{i\sqrt{3}\omega_0 t} \\ C_1 e^{i\omega t} - C_2 e^{i\sqrt{3}\omega_0 t} \end{bmatrix}$$

Another example: sol'ns for periodic f'ns $x : [0, L]$

→ Free particles in QM: $|n\rangle = e^{inx}$ define states

momentum operator: $p \propto \frac{d}{dx} \rightarrow p|n\rangle = in|n\rangle$ Not Hermitian

generic operator: $D = -i \frac{d}{dx}$ → is hermitian

$\langle f | D | g \rangle = -i \int f^* g' \, dx$ $\quad \frac{d}{dx}[g]$

$f \rightarrow f^*$

$-i \rightarrow i$

does it equal (hermitian)?

$g^* \rightarrow g$

$\langle g | D^+ | f \rangle^* = \left[-i \int g^* f' \, dx \right]^* = i \int g \frac{df^*}{dx} \cdot d =$

$= i \int g \, df^*$ $\quad u \quad v$

integration by parts

$$= i\left[\underbrace{g f^* \Big|_0^2}_{0 \text{ for periodic}} - \int f^* dg\right] = -i \int f^* \overbrace{g' dx}^{dg}$$

Commentary #5

"S I R" eq^{n}'s for epidemics

Nonlinear "SIR"

population (fixed) $\quad P = \underset{S(t)}{\#\,\text{Susceptible}} + \underset{I(t)}{\#\,\text{Infected}} + \underset{R(t)}{\#\,\text{Recovered}}$

commingling theory

each I-person has n contacts per day

$$\#\text{ new } I/\text{day} = \frac{n}{P} \cdot S$$

$$\dot{S} = -\frac{n}{P} S I$$

$$\dot{R} = r I$$

$$P = \text{constant} \qquad \dot{P} = 0 = \dot{S} + \dot{I} + \dot{R}$$

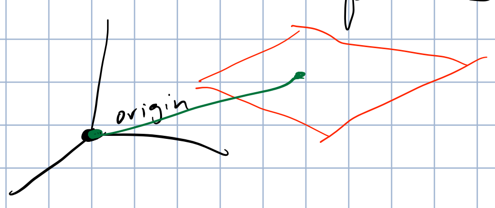
$$\longrightarrow \quad \dot{I} = -\dot{S} - \dot{R}$$

$$\dot{S} = -a\, S I \longrightarrow \text{Non-linear!}$$
$$\dot{I} = a S I - r I$$
$$\dot{R} = r I$$

to linearize:

$$\left.\begin{aligned} \dot{S} &= -aS \\ \dot{I} &= aS - rI \\ \dot{R} &= rI \end{aligned}\right\} \text{not realistic}$$

Next Time: extrema problems



Plane: $ax + by + cz + d = 0$

Shortest dist. from origin to plane

$$\longrightarrow \sqrt{x^2 + y^2 + z^2}$$

↳ x, y, z not indpt. need to be on plane