Convolution cont'd

last time: $P(x) * Q(x) = \int_{-\infty}^{\infty} P(x-x')\,Q(x')\,dx'$

can show $\widetilde{P \cdot Q} = \frac{1}{\sqrt{2\pi}} \cdot \tilde{P}(\omega) * \tilde{Q}(\omega)$

notice if I change variable: $u = x - x'$ $\quad dx' = -du$

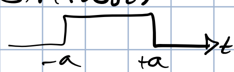
$$P * Q = \int_{+\infty}^{-\infty} P(u)\,Q(x-u)\,(-du)$$

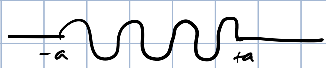
$$= \int_{-\infty}^{\infty} P(u)\,Q(x-u)\,du$$

$$= Q * P$$

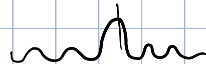
apply to "radio problem"

$P = \sin(\omega_0 t)$

$Q =$  $=$ "boxcar $f^{\underline{n}}$" $= \mathrm{rect}(t/a)$

$\rightarrow P(t) \cdot Q(t) =$ 

already did: $\tilde{P}(\omega) = \sqrt{\frac{\pi}{2}}\, i \left(\delta(\omega+\omega_0) - \delta(\omega-\omega_0) \right)$

$\tilde{Q}(\omega) = \frac{2a}{\sqrt{2\pi}} \frac{\sin(\omega a)}{\omega a}$ 

$$\widetilde{PQ} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{P}(\omega-\omega')\,\tilde{Q}(\omega')\,d\omega'$$

or $$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{Q}(\omega-\omega')\,\tilde{P}(\omega')\,d\omega'$$ → eg b/c Phas δ

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{Q}(\omega-\omega') \cdot \sqrt{\frac{\pi}{2}}\, i \left(\delta(\omega'+\omega_0) - \delta(\omega'-\omega_0)\right)$$

$\omega' = -\omega_0$ $\qquad \omega' = \omega_0$

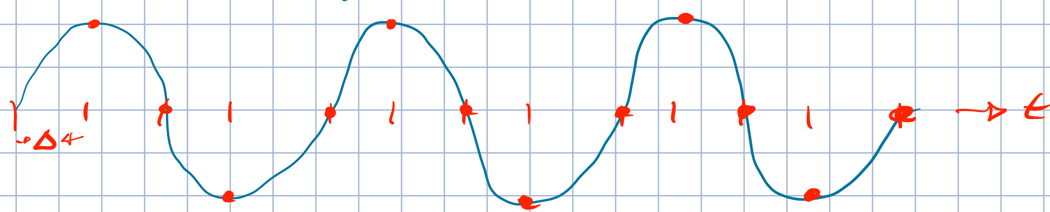
want to have $\delta(0)$

$$= \frac{1}{\sqrt{2\pi}} \cdot \sqrt{\frac{\pi}{2}}\, i\, \frac{2a}{\sqrt{2\pi}} \left[\frac{\sin[(\omega+\omega_0)a]}{(\omega+\omega_0)a} - \frac{\sin[(\omega-\omega_0)a]}{(\omega-\omega_0)a} \right]$$

Convolution ⟶ Shannon's th^{m} (Nyquist's Th^{m})

Suppose I sample sin wave every $\Delta t \equiv \Delta$ (like every second)

What is max. freq. I can reconstruct?



data can I reconstruct solid line?

Measurements can be seen as a convolution

$P(t) * Q(t)$

$P(t) = \sin(\omega_0 t)$

$Q(t) =$ "Dirac comb" $= \sum_{n=-\infty}^{\infty} \delta(t - n\cdot\Delta)$

$t' = t - n\Delta$

$$P * Q = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt' \sin(\omega_0 t') \sum \delta(t - n\Delta - t') = \frac{1}{\sqrt{2\pi}} \sum_n \int_{-\infty}^{\infty} dt' \sin(\omega_0 t')\, \delta(t - n\Delta - t')$$

$$= \frac{1}{\sqrt{2\pi}} \sum_n \sin[\omega_0 (t - n\Delta)]$$

now transform to get spectrum from data

can be done b/c we know $\tilde{P}$ & $\tilde{Q}$

$$\tilde{P}(\omega) \propto [\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$$

$$\tilde{Q}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, e^{-i\omega t} \left[\sum_n \delta(t - n\Delta)\right] = \frac{1}{\sqrt{2\pi}} \sum_n e^{-in\Delta\omega}$$

dummy variable always comes after (−) in convolution

$$\widetilde{PQ}(\omega) \propto \int_{-\infty}^{\infty} \tilde{P}(\omega')\, \tilde{Q}(\omega - \omega')\, d\omega'$$

$$= \sum_n \int_{-\infty}^{\infty} [\delta(\omega' + \omega_0) - \delta(\omega' - \omega_0)]\, e^{-in\Delta(\omega - \omega')}\, d\omega'$$

$$= \sum_n \left[e^{-in\Delta(\omega + \omega_0)} - e^{-in\Delta(\omega - \omega_0)} \right]$$

$$= \sum_n e^{-in\Delta\omega} \left[e^{-in\Delta\omega_0} - e^{in\Delta\omega_0} \right]$$

$$= \sum_n \underbrace{e^{-in\Delta\omega}}_{\text{where } \omega \text{ is } @} \cdot \left(-2i \sin(n\Delta\omega_0) \right)$$

notice: get same value for $\omega \to \omega + \frac{2\pi}{n\Delta}$

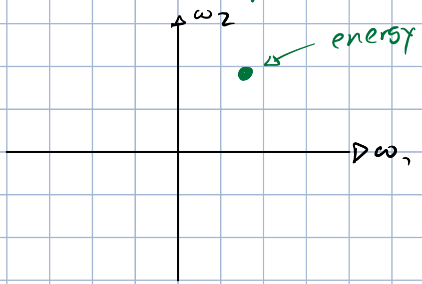
$\widetilde{PQ}(\omega)$ periodic in $\frac{2\pi}{n\Delta}$ ⟶ "aliasing"

therefore $\Delta < \frac{\pi}{\omega_{max}}$ or $\omega_{sampling} = \frac{2\pi}{\Delta} > 2\,\omega_{max}$

Image Processing

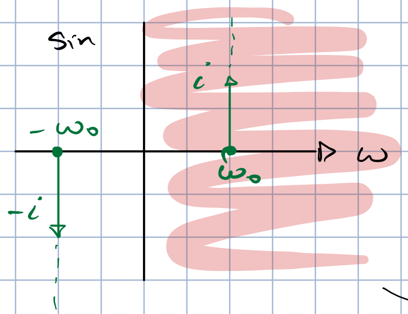
F.T. in 2-D $\quad \tilde{F}(\omega_1,\omega_2) = \left(\frac{1}{\sqrt{2\pi}}\right)^2 \cdot \int_{-\infty}^{\infty} dx\, e^{-i\omega_1 x} \int_{-\infty}^{\infty} dy\, e^{-i\omega_2 y}\, F(x,y)$

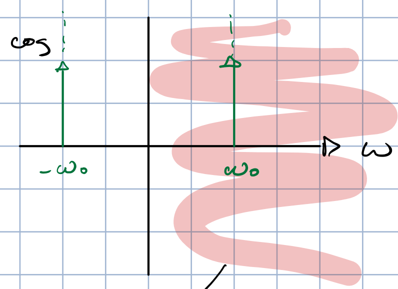
$\tilde{F}$ often expressed as power map : $\left|\tilde{F}(\omega_1,\omega_2)\right|^2$ → energy in freq. band



negative ω → $\widetilde{\sin(\omega_0 t)} = \sqrt{\frac{\pi}{2}}\, i \left[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right]$

$\widetilde{\cos(\omega_0 t)} = \sqrt{\frac{\pi}{2}} \left[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right]$





need negative ω to figure out whether working w/ sin or cos

if only $\omega > 0$ can't distinguish