Dirac δ Function in 1-D: $\int_{-\infty}^{\infty} f(x)\,\delta(x-x_0)\,dx = f(x_0)$

introduced as $\delta(x_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x_0)}\,dk$

Consider Gaussian $f^{\underline{n}}$, keep normalization constant

$e^{-n^2x^2}$; know $\int_{-\infty}^{\infty} e^{-n^2x^2}dx = \frac{\sqrt{\pi}}{n}$

$\frac{n}{\sqrt{\pi}}\,e^{-n^2x^2}$ integrates as 1 for any n

@ full width half max.,  $= 2.35\,\sigma = \frac{2.35}{\sqrt{2}\,n} \to 0$ for $n\to\infty$

$\to$ height $\to\infty$ as $n\to\infty$

Consider $\int_{-\infty}^{\infty} dx\, f(x) \underbrace{\frac{n}{\sqrt{\pi}}\,e^{-n^2x^2}}_{\delta_n(x)}$ as $n\to\infty$

if true, only contributions are for x near 0

Taylor expand: $f(x) \cong f(0) + x\, f'(x)\big|_{x=0}$

$\to \int_{-\infty}^{\infty} dx\, f(x)\,\delta_n(x) = f(x) \underbrace{\int_{-\infty}^{\infty} \delta_n(x)dx}_{1} + f'(x)\big|_{x=0} \cdot \int_{-\infty}^{\infty} x\,\delta_n(x)dx$

$\int_{-\infty}^{\infty} dx\; x\,\delta_n(x) \underset{u\equiv x^2}{=} \frac{1}{2}\underbrace{2\int_0^{\infty}}_{} \frac{n}{\sqrt{\pi}}\,e^{-n^2u}\,du = \frac{n}{\sqrt{\pi}}\cdot\frac{1}{-n^2}\underbrace{e^{-n^2u}\Big|_0^{\infty}}_{} \to 0$ as $n\to\infty$

$\to$ goes twice over same area b/c even $f^{\underline{n}}$

$\to$ 0 @ ∞ & 1 @ 0

$\to \int_{-\infty}^{\infty} dx\, f(x)\,\delta_n(x) \xrightarrow[n\to\infty]{} f(0)$

Solving DE's w/ F.T.

Wave eq$^{\underline{n}}$: $u''(x,t) = \frac{1}{v^2}\,\ddot{u}(x,t)$ w/ I.C. $u(x,0) \equiv f(x)$

take F.T. wrt. x: $(ik)^2\,\tilde{u}(k,t) = \frac{1}{\sqrt{2\pi}}\int u''\,e^{ikx}dk = \frac{1}{v^2}\,\tilde{\ddot{u}}(k,t)$

b/c F.T. wrt x, order for this doesn't matter

get $\frac{\partial^2}{\partial t^2}\left(\tilde{u}(k,t)\right) = -v^2k^2\,\tilde{u}(k,t)$ $\quad \ddot{\tilde{u}}(k,t)$

$\to \tilde{u} \propto e^{\pm ivkt}$

F.T. $f(x)$

Using I.C. $\tilde{u}(k,0) = \tilde{f}(k) \to \tilde{u}(k,t) = \tilde{f}(k)\cdot e^{\pm ivkt}$

inverse F.T.

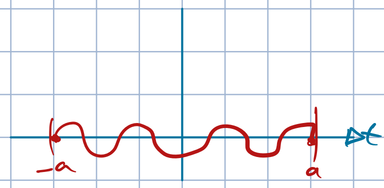
$u(x,t) = \frac{1}{\sqrt{2\pi}}\int e^{-ikx}dk\;\tilde{u}(k,t)$

$= \frac{1}{\sqrt{2\pi}}\int dk\; e^{-ikx}\cdot\tilde{f}(k)\cdot e^{\pm ivkt}$

$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\; e^{-ik(x\mp vt)}\,\tilde{f}(k) = f(x\mp vt)$

Uncertainty Principle Revisited

"Radio Problem" : have a source $f(t) = \sin(\omega_0 t)$



Practical Problem : can only analyze for fixed amt. time $\rightarrow$ during $-a < t < a$

get frequency content by F.T.

rewrite sin as exp's

$$c(\omega) = \tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \sin(\omega_0 t)\, e^{i\omega t}\, dt = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{2i} \int \underline{\left(e^{i\omega_0 t} - e^{-i\omega_0 t}\right)} e^{i\omega t}\, dt$$

$$= -\frac{i}{2\sqrt{2\pi}} \int_{-a}^{a} e^{i(\omega+\omega_0)t} - e^{i(\omega-\omega_0)t}\, dt$$

$$= -\frac{i}{2\sqrt{2\pi}} \left\{ \frac{e^{i(\omega+\omega_0)t}}{i(\omega+\omega_0)}\Bigg|_{-a}^{a} - \frac{e^{i(\omega-\omega_0)t}}{i(\omega-\omega_0)}\Bigg|_{-a}^{a} \right\}$$

rewrite exp as sin's

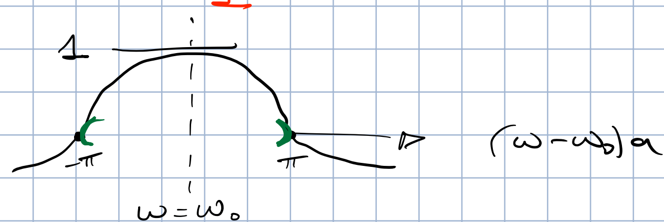
$2i$

$$= \frac{-1}{2\sqrt{2\pi}} \cdot \left\{ \frac{\sin[(\omega_0+\omega)\cdot a]}{\omega+\omega_0} - \underbrace{\frac{\sin[(\omega-\omega_0)a]}{\omega-\omega_0}} \right\}$$

$\rightarrow$ dominates when $\omega \approx \omega_0$

$\rightarrow$ 1

"width" $= 2\pi$ in $(\omega-\omega_0)a$ space



interpret $(\omega-\omega_0)a \sim [-\pi, \pi]$ as uncertainty in ω

$\rightarrow$ $\Delta\omega = \omega(\pi) - \omega(-\pi) = \frac{2\pi}{a}$

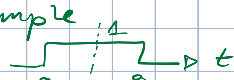
by def^{n} $\Delta t = 2a$

$\longrightarrow$ $\Delta\omega\, \Delta t = \frac{2\pi}{a} \cdot 2a = 4\pi$

"uncertainty principle" for waves

Convolution

$f(t) =$  in last example

$=$  $*$ 

$\rightarrow$ only look @ period we measured

consider $f(t) = P(t) * Q(t)$

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, e^{-i\omega t} \{P(t) * Q(t)\}$$

dummy variable

$$Q(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega'\, e^{+i\omega' t}\, \tilde{Q}(\omega')$$

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} d\omega' \, \tilde{Q}(\omega') \cdot \underbrace{\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \; P(t) \cdot e^{-i(\omega-\omega')t} \right)}_{\text{F.T. of } P(t) \,\to\, \tilde{P}(\omega-\omega')}$$

$$\tilde{f}(\omega) \equiv \widetilde{PQ} = \frac{1}{2\pi} \underbrace{\int_{-\infty}^{\infty} d\omega' \, \tilde{Q}(\omega') \, \tilde{P}(\omega - \omega_0)}_{\equiv \tilde{Q} * \tilde{P}} \longrightarrow \text{convolution}$$

↳ star operator, not multiplication

likewise, if $\tilde{f}(\omega) = \tilde{P}(\omega) \; \tilde{Q}(\omega)$

$$\longrightarrow \quad f(t) = \frac{1}{\sqrt{2\pi}} \cdot \quad Q * P \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt' \; Q(t') \, P(t-t')$$