From L1a, w/periodic f^k in $[-L, L]$, let $L \to \infty$

$$f(x) = \sum_{-\infty}^{\infty} C_n e^{ik_n x} \xrightarrow{k \to \infty} \frac{\alpha}{\pi} \int_{-\infty}^{\infty} c(k)\, e^{ikx} dx$$

$$C_n = \frac{1}{2\alpha} \int_{-\infty}^{\infty} e^{-ik_n x} f(x)\, dx \longrightarrow C(k) = \frac{1}{2\alpha} \int_{-\infty}^{\infty} f(x)\, e^{-ikx} dx$$

α is established as convention, in RHB: $\frac{\alpha}{\pi} = \frac{1}{2\alpha} = \frac{1}{\sqrt{2\pi}}$; others use $\frac{1}{2\pi}$ & 1

not same

given $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, e^{ikx} c(k)$

then $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, f(x)\, e^{-i\tilde{k}x} = c(\tilde{k})$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-i\tilde{k}x} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, e^{ikx} c(k) \right)$$

$$= \int_{-\infty}^{\infty} dk\, c(k) \underbrace{\int_{-\infty}^{\infty} dx\, e^{i(k-\tilde{k})x} \frac{1}{2\pi}}_{\equiv \delta(k-\tilde{k})}$$

$\to$ dirac delta "f^k"

$$= c(\tilde{k})$$

in general, $\int_{-\infty}^{\infty} d\tilde{k}\, \delta(k-\tilde{k})\, g(\tilde{k}) = g(k)$

note $\int_{-\infty}^{\infty} \delta(k)\, dk = 1$

Suppose $f(t) = \sin(\omega_0 t)$

$$= \frac{1}{2i}\left(e^{i\omega_0 t} - e^{-i\omega_0 t}\right)$$

now take Fourier XFM

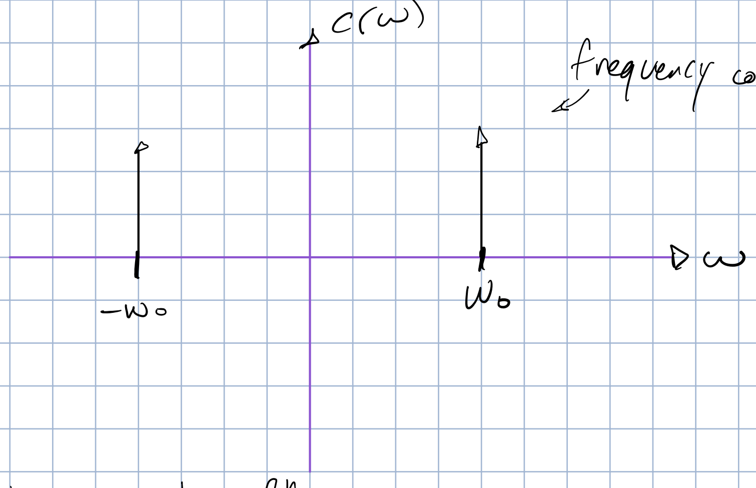
$$c(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, f(t)\, e^{-i\omega t}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{2i} \int_{-\infty}^{\infty} dt \left[e^{-i(\omega-\omega_0)t} - e^{-i(\omega+\omega_0)t} \right]$$

now using $\delta(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} dt$

$$c(\omega) = \frac{1}{\sqrt{2\pi}} \frac{1}{2i} \left\{ \underbrace{\int_{+\infty}^{-\infty} e^{i(\omega-\omega_0)t} dt}_{2\pi\,\delta(\omega-\omega_0)} - \underbrace{\int_{+\infty}^{-\infty} e^{i(\omega+\omega_0)t} dt}_{2\pi\,\delta(\omega+\omega_0)} \right\}$$

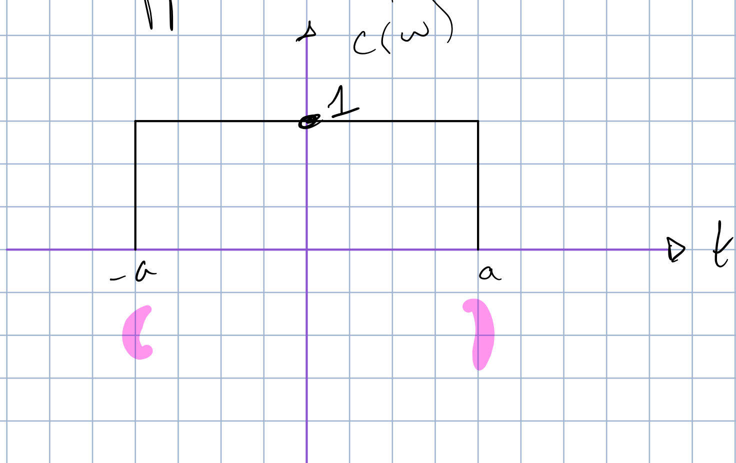
each has form of dirac delta f^k



frequency content of f^n @ $-\omega_0$ & ω_0

$C(\omega)$ is zero everywhere except @ $\pm\omega_0$

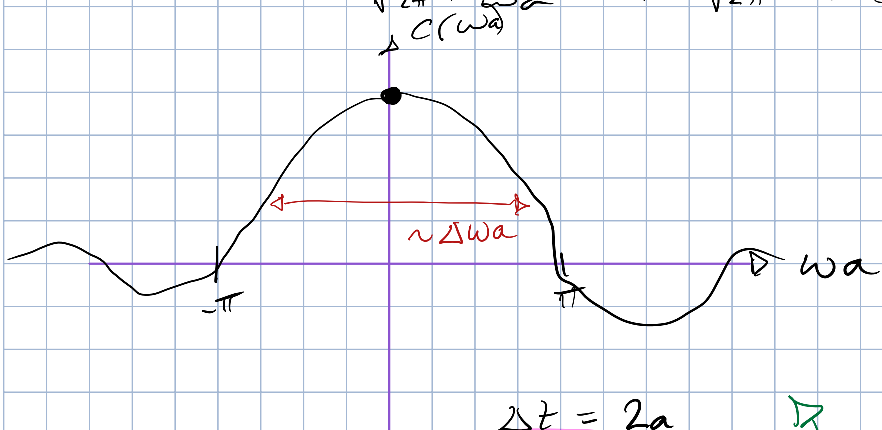
now suppose a box f^n



$$C(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, f(t)\, e^{-i\omega t} = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} dt \cdot 1 \cdot e^{-i\omega t}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{-i\omega} e^{-i\omega t}\Big|_{-a}^{a} = \frac{1}{\sqrt{2\pi}} \frac{1}{-i\omega} \left(e^{i\omega a} - e^{-i\omega a} \right)$$

$$= \frac{2a \sin(\omega a)}{\sqrt{2\pi} \cdot \omega a} = \frac{1}{\sqrt{2\pi}} \frac{1}{\omega} \sin(\omega a)$$



$\Delta t = 2a$

have ability to make sample of something in $2a$, I'll get measurement of frequencies composing square box

Longer examination time, more defined frequency

$$\Delta\omega\, \Delta t = \frac{2\pi}{a} \cdot 2a = 4\pi$$

look @ time domain or frequency domain w/ Fourier Transform pair

Fourier Transform of a derivative

F.T. of $f(t) = \tilde{f}(\omega) \equiv c(\omega)$

$\frac{d}{dt}[f(t)] = \dot{f}(t)$

$\tilde{\dot{f}}(\omega) = \int_{-\infty}^{\infty} dt \; e^{-i\omega t} \frac{df}{dt} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} df$? integration by parts

$= \frac{1}{\sqrt{2\pi}} \left[f \cdot e^{-i\omega t} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(t) \, d(e^{-i\omega t}) \right]$

oscillates

$= 0$ as long as $f \to 0$ @ $\pm\infty$ (localized f^n)

$= \frac{1}{\sqrt{2\pi}} \cdot - \left(\int_{-\infty}^{\infty} f(t) \cdot -i\omega e^{-i\omega t} dt \right)$

$= \frac{1}{\sqrt{2\pi}} \cdot i\omega \int_{-\infty}^{\infty} f(t) \, e^{-i\omega t} dt$

$\tilde{\dot{f}}(\omega) = i\omega \; \tilde{f}(t)$

f^n of ω?

Consider ODE $\ddot{f}(t) + \alpha^2 f(t) = Q(t)$ take $t \to \omega$

$\ddot{f}$ transforms: $\tilde{\ddot{f}}(\omega)$

$\tilde{\ddot{f}} + \alpha^2 \tilde{f} = \tilde{Q}$

$(i\omega)(i\omega)\tilde{f} + \alpha^2 \tilde{f} = \tilde{Q}$ $\qquad$ $f(\omega)$ same as $c(\omega)$

$\tilde{f}(\omega) = \frac{\tilde{Q}(\omega)}{\alpha^2 - \omega^2}$

transform back: $f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \; e^{i\omega t} \tilde{f}(\omega)$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \; e^{i\omega t} \frac{\tilde{Q}(\omega)}{\alpha^2 - \omega^2}$

go back to t

if $Q(t) = A\cos(\omega_0 t)$,

$\tilde{Q}(\omega) = \frac{A}{\sqrt{2\pi}} [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$

$f(t) = \frac{1}{2(\sqrt{2\pi})^2} A \int_{-\infty}^{\infty} d\omega \; e^{i\omega t} \frac{\delta(\omega - \omega_0) + \delta(\omega + \omega_0)}{\alpha^2 - \omega^2}$

$= \frac{A}{4\pi} \frac{1}{\alpha^2 - \omega^2} [e^{i\omega_0 t} + e^{-i\omega_0 t}]$

$= \frac{A}{2\pi} \frac{\cos(\omega_0 t)}{\alpha^2 - \omega_0^2}$