LVS stuff

Consider LVS of $f(x) = f(x+2L)$ on $-L < x < L$

2 Hermitian operators: $-\frac{d^2}{dx^2}$ & $-i\frac{d}{dx}$

$-\frac{d^2}{dx^2}$: $-f'' = \lambda^2 f \longrightarrow e^{i\lambda x}$

$-i\frac{d}{dx}$: $-if' = \lambda f \longrightarrow e^{i\lambda x}$

$f(x+2L) = e^{2L\lambda i} e^{i\lambda x}$

$= e^{2L\lambda i} f(x)$

$= 1\, f(x)$

$\Rightarrow \lambda = \frac{\pi}{L} \cdot n$

$\Rightarrow f(x) = \sum_{-\infty}^{\infty} c_n |n\rangle = \sum c_n e^{(\pi n i x)/L} = \sum c_n e^{ik_n x}$ $\quad k_n \equiv (\pi \cdot n)/L$ "wavenumber"

$\langle m | n \rangle = \int_{-L}^{L} dx\, e^{-ik_m x} e^{ik_n x} = \int_{-L}^{L} dx\, e^{\frac{i\pi}{L}(n-m)x} = 2L\,\delta_{nm}$

so $|n\rangle$ are orthogonal

to normalize instead of $\frac{1}{\sqrt{2L}}$, use $|\hat{n}\rangle = \frac{\alpha}{L} e^{ik_n x}$

now $\hat{f}(x) = \frac{\alpha}{L} \sum c_n e^{ik_n x} = \sum c_n |\hat{n}\rangle$

find c_n:

$\langle m | f \rangle = \frac{\alpha}{L} \cdot \int_{-L}^{L} dx\, e^{-ik_m x} f(x) =$

$\langle m | f \rangle$

$\langle m | f \rangle \langle \hat{n} | \hat{n} \rangle$

$\langle m | \hat{n} \rangle \langle \hat{n} | f \rangle$

$\frac{\alpha}{L} \langle m | n \rangle \cdot \frac{\alpha}{L} \langle n | f \rangle$

$\Rightarrow \frac{\alpha^2}{L^2} \langle m | n \rangle \langle n | f \rangle$

$\langle n | f \rangle = \langle n | \sum_{-\infty}^{\infty} c_n | n \rangle = \sum_{-\infty}^{\infty} c_n \langle n | n \rangle$

$= \sum_{-\infty}^{\infty} c_n$

$= \frac{\alpha}{L} \int_{-L}^{L} dx\, e^{-ik_m x} \frac{\alpha}{L} \sum c_n e^{ik_n x}$

$= \frac{\alpha^2}{L^2} \sum_{m=1}^{\infty} c_n \cdot \int_{-L}^{L} e^{-ik_m x} e^{ik_n x} dx$

$= \frac{\alpha^2}{L^2} \cdot \sum_{m=1}^{\infty} c_n \langle m | n \rangle$

$= \frac{\alpha^2}{L^2} \cdot \sum c_n\, 2L\,\delta_{nm} = \frac{2\alpha^2}{L} c_m$

$\Rightarrow \frac{\alpha^2}{L^2} (2L\,\delta_{nm}) (\sum_{-\infty}^{\infty} c_n)$

$\frac{2\alpha^2}{L} \sum_{-\infty}^{\infty} c_n \delta_{nm} = \frac{2\alpha^2}{L} c_m = \langle m | f \rangle \Rightarrow c_m = \frac{L}{2\alpha^2} \cdot \langle m | f \rangle$

Rewrite: $L \to \infty$ $\quad c_m = \frac{1}{2\alpha} \int_{-\infty}^{\infty} e^{-ik_m x} f(x)\, dx$

Notice: $k_{n+1} - k_n = \frac{\pi}{L}(n+1 - n) = \frac{\pi}{L} \equiv dk$

"step size"

$$\longrightarrow \quad f = \sum_n c_n \left(\frac{\alpha}{L} e^{ik_n x} \right) \left(dk \cdot \underbrace{\frac{L}{\pi}}_{=1} \right)$$

$$= \frac{\alpha}{\pi} \sum c_n e^{ik_n x} dk \xrightarrow{L \to \infty} \frac{\alpha}{\pi} \int_{-\infty}^{\infty} c(k) e^{ikx} dx$$

k_n gets smaller as $L \to \infty$

Fourier pair

For $L \to \infty$:

$$f(x) = \frac{\alpha}{\pi} \int c(k)\, e^{ikx} dk \qquad c(k) = \frac{1}{2\alpha} \int f(x)\, e^{-ikx} dx$$

	$\frac{\alpha}{\pi}$	$\frac{1}{2\alpha}$
$\alpha = \sqrt{\pi/2}$	$1/\sqrt{2\pi}$	$1/\sqrt{2\pi}$
$\alpha = 1/2$	$1/2\pi$	1
$\alpha = \pi$	1	$1/2\pi$

conventions for Fourier Transform

can expand $f(x)$ for $-\infty < x < \infty$
by finding eigenvalues & take continuum limit

$\mathcal{L}$: $-\frac{d^2}{dx^2}$, $i\,d/dx$ are Hermitian

Form $\langle n | \mathcal{L} | m \rangle$ $\qquad |n\rangle = e^{(i\pi n x)/L}$

$-\frac{d^2}{dx^2}$: $\displaystyle \int dx\, e^{(-n\pi i x)/L} \left(-\frac{d^2}{dx^2} \left(e^{(i\pi m x)/L} \right) \right)$

$$= \int dx\; e^{-\frac{n\pi i x}{L}} \cdot -\left(\frac{i\pi m}{L}\right)^2 \cdot e^{\frac{i\pi m x}{L}}$$

$$= \left(\frac{\pi m}{L}\right)^2 \int dx\; e^{\frac{-in\pi x}{L}}\, e^{\frac{i\pi m x}{L}}$$

$$= \left(\frac{\pi m}{L}\right)^2 \int_{-L}^{L} dx\; e^{i(\pi x)(m-n)/L} = 2L \cdot \left(\frac{\pi m}{L}\right)^2 \delta_{nm} = \frac{2\pi^2 m^2}{L} \delta_{nm}$$

$-i\,d/dx$: $\displaystyle \int dx\, e^{-\frac{n\pi i x}{L}} \cdot -i \frac{d}{dx}\left[e^{\frac{i\pi m x}{L}} \right]$

$$= \int dx\, e^{-\frac{\pi n i x}{L}} \cdot \frac{\pi m}{L}\, e^{\frac{i\pi m x}{L}}$$

$$= \frac{\cancel{\pi m}}{2} \int dx \; e^{ix\pi(n-m)} = \frac{\pi m}{L} \cdot 2L \, \delta_{nm} = 2\pi m \, \delta_{nm}$$

Both cases: $\mathcal{L}_{nm} = \langle n | \mathcal{L} | m \rangle = \mathcal{L}_{mn} = (\mathcal{L}^T)_{nm}$ order doesn't matter

$\rightarrow$ both hermitian

Same as $\langle n | \left(\mathcal{L} | m \rangle \right) = \left(\langle m | \mathcal{L}^+ | n \rangle^* \right)$

flip, need new $\mathcal{L}$

how do write s.t. $\mathcal{L}$ operating on $|n\rangle$ rather than $|m\rangle$ w/a related $\mathcal{L}$

$\rightarrow$ call it $\mathcal{L}^+$

if I form $\mathcal{L}_{nm}$ & $\mathcal{L}_{mn}^+$ matrices w/operators $-\frac{d^2}{dx^2}$ or $-i\frac{d}{dx}$

get $\mathcal{L}_{nm} = \mathcal{L}_{mn}^* = (\mathcal{L}^+)_{nm}$

Adjoint def$^{\underline{n}}$

$\mathcal{L} | w \rangle = | z \rangle$

$\langle v | \mathcal{L} | w \rangle = \langle v | z \rangle$

$\langle v | z \rangle^* = \langle z | v \rangle$ what's z?

$= \langle w | \mathcal{L}^+ | v \rangle = \langle v | \mathcal{L} | w \rangle^*$

v & $w \rightarrow$ basis vectors find $[L^+]_{nm} = L_{mn}^* = [L^{T*}]_{nm}$

hermitian cases: $\mathcal{L}^+ = \mathcal{L}$ orthogonal basis

unitary cases: $\mathcal{L}^{-1} = \mathcal{L}^+$ basis rotations