defⁿ impt. general examples titles solution methods

in most specific applications, necessary to choose a coord. system & break down to components

$$\vec{a} = a_x \hat{i} + a_y \hat{j} + a_z \hat{z}$$

can have arbitrary # of dimensions

8.1 Vector Spaces

set of objects (vectors) $\vec{a}, \vec{b}, \vec{c}$ form a linear vector space (LVS) if...

① set is closed under commutative & associative addition

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$$

② set is closed under multiplication by a scalar to form a new vector $\lambda\vec{a}$, both operations being both distributive & associative

$$\lambda(\vec{a} + \vec{b}) = \lambda\vec{a} + \lambda\vec{b}$$

$$(\lambda + \mu)\vec{a} = \lambda\vec{a} + \mu\vec{a}$$

$$\lambda(\mu\vec{a}) = \mu(\lambda\vec{a})$$

λ, μ are arbitrary scalars

③ there exists a null vector s.t.

$$\vec{a} + \vec{0} = \vec{a} \quad \text{for all } \vec{a}$$

④ multiplication by unity leaves any vector unchanged

$$1 \times \vec{a} = \vec{a}$$

⑤ all vectors have a corresponding negative vector s.t.

$$\vec{a} + (-\vec{a}) = \vec{0}$$

if we restrict all scalars to be real, we obtain a real vector space

space ≈ vector space ≈ LVS

span of a set of vectors $\vec{a}, \vec{b}, \ldots, \vec{s}$ defined as set of all vectors that may be written as linear sum of original

$$\vec{x} = \alpha \vec{a} + \beta \vec{b} + \ldots + \gamma \vec{s}$$

if $\vec{x} = \vec{0}$ $\rightarrow$ $\alpha \vec{a} + \beta \vec{b} + \ldots + \gamma \vec{s} = \vec{0}$

$\rightarrow$ set of vectors $\vec{a}, \vec{b}, \ldots, \vec{s}$ are **linearly dependent**

at least one vector is redundant. can be written in terms of another vector

if it is not satisfied by any set of coefficients, vectors are **linearly indpt**

if there exists N linearly indpt. vectors, but no set of $N+1$ linearly indpt. vectors, LVS said to be **N-dimensional**

Basis Vectors

if V is an N-dimensional LVS, any set of N linearly indpt. vectors $\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_N$ forms a basis for V

if $\vec{x}$ is an arbitrary vector lying in V, then set of $N+1$ vectors $(\vec{x}, \hat{e}_1, \hat{e}_2, \ldots, \hat{e}_N)$ must be linearly dependent

$$\rightarrow \quad \alpha \hat{e}_1 + \beta \hat{e}_2 + \ldots + \sigma e_N + \chi x = 0$$

where $\alpha, \beta, \sigma, \chi \neq 0$

may rewrite x as a linear sum of vectors $\hat{e}_i$

$$\vec{x} = x_1 \hat{e}_1 + x_2 \hat{e}_2 + \ldots + x_N \hat{e}_N = \sum_{i=1}^{N} x_i \hat{e}_i$$

$x_1 = -\frac{\alpha}{\chi}$

$x_2 = -\frac{\beta}{\chi}$

b/c $\vec{x}$ lying in span of V can be expressed in terms of base vectors $\hat{e}_i$, said to be a **complete set**

coefficients x_i are components of $\vec{x}$ wrt $\hat{e}_i$ bases. these components are unique

since if both $\vec{x} = \sum_{i=1}^{N} x_i \hat{e}_i$ & $\vec{x} = \sum_{i=1}^{N} x_i \hat{e}_i$

then $\sum_{i=1}^{N} (x_i - y_i)\hat{e}_i = \vec{0}$ only sol^n: $x_i = y_i$ for all $i = 1, 2, \ldots, N$

can see that any set of N linearly indpt. vectors can form basis for N-dimensional LVS

if we choose a different e_i', $\forall i=1,2,\dots,N$, can write $\vec{x}$...

$$\vec{x} = x_1'\hat{e}_1' + x_2'\hat{e}_2' + \dots x_n'\hat{e}_N' = \sum_{i=1}^{N} x_i'\hat{e}_i'$$

vector $\vec{x}$ is indpt. of basis

Inner Product

generalization of dot product to more abstract LVS

$$\langle \vec{a} | \vec{b} \rangle = \langle \vec{b} | \vec{a} \rangle^*$$ → complex conjugate is $*$

$$\langle \vec{a} | (\lambda\vec{b} + \mu\vec{c}) \rangle = \lambda \langle \vec{a} | \vec{b} \rangle + \mu \langle \vec{a} | \vec{c} \rangle$$

$\langle \vec{a} | \lambda\vec{b} \rangle + \langle \vec{a} | \mu\vec{c} \rangle$

$$\longrightarrow \langle (\lambda\vec{a} + \mu\vec{b}) | \vec{c} \rangle = \langle \lambda\vec{a} | \vec{c} \rangle + \langle \mu\vec{b} | \vec{c} \rangle$$
$$= \lambda^* \langle \vec{a} | \vec{c} \rangle + \mu^* \langle \vec{b} | \vec{c} \rangle$$

$$\langle \lambda\vec{a} + \mu\vec{b} \rangle = \lambda^*\mu \langle \vec{a} + \vec{b} \rangle$$

2 vectors in a LVS are orthogonal if $\langle \vec{a} | \vec{b} \rangle = 0$

introduce into our N-dimensional vector space a basis $\hat{e}_1, \hat{e}_2, \dots, \hat{e}_N$ having property of being orthonormal - basis vectors are mutually orthogonal & each has a unit norm

$$\longrightarrow \langle \hat{e}_i | \hat{e}_j \rangle = \delta_{ij}$$

$$\delta_{ij} = \begin{cases} 1 & \text{for } i=j \\ 0 & \text{for } i \neq j \end{cases}$$

may express any 2 vectors:

$$\vec{a} = \sum_{i=1}^{N} a_i\hat{e}_i \qquad \vec{b} = \sum_{i=1}^{N} b_i\hat{e}_i$$

in an orthonormal basis,

$$\langle \hat{e}_j | \vec{a} \rangle = \sum_{i=1}^{N} \langle \hat{e}_j | a_i\hat{e}_i \rangle = \sum_{i=1}^{N} a_i \langle \hat{e}_j | \hat{e}_i \rangle = a_j$$

can write inner product of $\vec{a}$ & $\vec{b}$ in terms of components in an orthonormal basis

$$\langle \vec{a} | \vec{b} \rangle = \langle a_1\hat{e}_1 + a_2\hat{e}_2 + \dots + a_N\hat{e}_N \,|\, b_1\hat{e}_1 + b_2\hat{e}_2 + \dots + b_N\hat{e}_N \rangle$$
$$= \sum_{i=1}^{N} a_i^* b_i \langle \hat{e}_i | \hat{e}_i \rangle + \sum_{i=1}^{N} \sum_{j \neq i}^{N} a_i^* b_j \langle \hat{e}_i | \hat{e}_j \rangle$$

$$= \sum_{i=1}^{N} a_i^* b_i$$

if not orthonormal & $\vec{a} = \sum_{i=1}^{N} a_i \hat{e}_i$ & $\vec{b} = \sum_{i=1}^{N} b_i \hat{e}_i$

$$\langle a | b \rangle = \langle \sum_{i=1}^{N} a_i \hat{e}_i \,|\, \sum_{j=1}^{N} b_j \hat{e}_j \rangle$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} a_i^* b_j \ \langle \hat{e}_i | \hat{e}_j \rangle$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} a_i^* G_{ij} b_j$$

require $G_{ij} = G_{ji}^*$

$\rightarrow \|\vec{a}\| = \langle \vec{a} | \vec{a} \rangle$ is real

$$\rightarrow \langle \vec{a} | \vec{a} \rangle^* = \sum_{i=1}^{N} \sum_{j=1}^{N} a_i \ G_{ij}^* a_j^*$$

$$= \sum_{j=1}^{N} \sum_{i=1}^{N} a_j^* \ G_{ji} \ a_i = \langle \vec{a}, \vec{a} \rangle$$

Some Useful Inequalities

Schwarz' inequality: $|\langle \vec{a} | \vec{b} \rangle| \leq \|\vec{a}\| \cdot \|\vec{b}\|$

Triangle inequality $\|\vec{a} + \vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$

Bessel's inequality requires orthonormal basis $\hat{e}_i$ $i = 1, 2, \ldots, N$

$$\|\vec{a}\|^2 \geq \sum_i |\langle \hat{e}_i | \vec{a} \rangle|^2$$

Parallelogram inequality $\|\vec{a} + \vec{b}\|^2 + \|\vec{a} - \vec{b}\|^2 = 2\left(\|\vec{a}\|^2 + \|\vec{b}\|^2\right)$

8.2 Linear Operators

a linear operator A associates w/ every vector $\vec{x}$ another vector $\vec{y} = A\vec{x}$

$$\rightarrow A(\lambda \vec{a} + \mu \vec{b}) = \lambda A \vec{a} + \mu A \vec{b}$$

$\rightarrow$ A 'operates' on $\vec{x}$ to give vector $\vec{y}$

if we introduce basis $\hat{e}_i$, action of A on each of the basis vectors to produce a linear combo. of latter:

$$A \hat{e}_j = \sum_{i=1}^{N} \underbrace{A_{ij}}_{\text{components of linear operator in } \hat{e}_i \text{ basis}} \hat{e}_i$$

$$\vec{y} = \sum_{i=1}^{N} y_i \hat{e}_i = A\left(\sum_{j=1}^{N} x_j \hat{e}_j\right) = \sum_{j=1}^{N} x_j \sum_{i=1}^{N} A_{ij} \hat{e}_i$$

$$\longrightarrow y_i = \sum_{j=1}^{N} A_{ij} x_j$$

if $\vec{y}$ belongs in a different LVS, which may be in general be M-dimensional, then analysis needs tweeking

introduce basis set $\hat{f}_i$ $\quad i = 1, 2, \ldots, M$

$$A\hat{e}_j = \sum_{i=1}^{M} A_{ij} \hat{f}_i$$ where components A_{ij} of A relate to both bases $\hat{e}_j$ & $\hat{f}_i$

Properties of Linear Operators

if $\vec{x}$ is a vector and A & B are 2 linear operators,

$$(A+B)\vec{x} = A\vec{x} + B\vec{x}$$

$$(\lambda A)\vec{x} = \lambda(A\vec{x})$$

$$(AB)\vec{x} = A(B\vec{x})$$

product of 2 linear operators is not in general commutative $\longrightarrow AB\vec{x} \neq BA\vec{x}$

can define null & identity operators:

$$O\vec{x} = \vec{0} \qquad I\vec{x} = \vec{x}$$

2 operators A & B are equal if $\quad A\vec{x} = B\vec{x} \quad$ for all $\vec{x}$

if there exists an operator A^{-1} s.t. $\quad AA^{-1} = A^{-1}A = I$

then A^{-1} is the inverse of A

8.3 Matrices

in particular basis $\hat{e}_i$ both vectors & linear operators can be described in terms of components w.r.t. bases

components can be seen as a matrix – array of nums.

if linear operator A transforms vectors from N dimensional LVS (w/ basis $\hat{e}_j$ $j = 1, 2, \ldots, N$) into an M-dimensional LVS (w/ basis $\hat{f}_i$ $i = 1, 2, \ldots, M$)

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1N} \end{bmatrix}$$

matrix elements A_{ij} are components of linear operator w.r.t. bases $\hat{e}_j$ + $\hat{f}_i$

$\longrightarrow A = \begin{bmatrix} A_{21} & A_{22} & \cdots & A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{M1} & A_{M2} & \cdots & A_{MN} \end{bmatrix}$

$$A_{ij} = (A)_{ij}$$

vector $\vec{x}$: basis $\hat{e}_i$ $i = 1, 2, \ldots, N$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$ ← column matrix

row matrix → $X = (x_1, x_2, \ldots x_N)^T$ transpose of a column matrix

different basis $\hat{e}_i'$ $$X' = \begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_N' \end{pmatrix}$$

→ all denote same vector $\vec{x}$ w/ different basis

8.4 Basic Matrix Algebra

consider 2 linear operators A & B operating on arbitrary vector $\vec{x}$

$$\sum_j (A+B)_{ij} x_j = \sum_j A_{ij} x_j + \sum_j B_{ij} x_j$$

$$\sum_j (\lambda A)_{ij} x_j = \lambda \sum_j A_{ij} x_j$$

$$\sum_j (AB)_{ij} x_j = \sum_k A_{ik} (Bx)_k = \sum_j \sum_k A_{ik} B_{kj} x_j$$

$$\longrightarrow (A+B)_{ij} = A_{ij} + B_{ij}$$

$$(\lambda A)_{ij} = \lambda A_{ij}$$

$$(AB)_{ij} = \sum_k A_{ik} B_{kj}$$

Matrix Addition and Multiplication by a Scalar

$$S_{ij} = A_{ij} + B_{ij}$$

for 2x3 matrix:

$$\begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{pmatrix} + \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \end{pmatrix}$$

$$= \begin{pmatrix} A_{11}+B_{11} & A_{12}+B_{12} & A_{13}+B_{13} \\ A_{21}+B_{21} & A_{22}+B_{22} & A_{23}+B_{23} \end{pmatrix}$$

must have same dimensions

$A + B = B + A$ $\qquad D = A - B$

matrix addition is associative & commutative

matrix multiplication is associative & distributive

$$\lambda \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{pmatrix} = \begin{pmatrix} \lambda A_{11} & \lambda A_{12} & A_{13} \\ \lambda A_{21} & \lambda A_{22} & \lambda A_{23} \end{pmatrix}$$

Multiplication of Matrices

consider transformation of 1 vector into another $\vec{y} = A\vec{x}$

components: $y_i = \sum_{j=1}^{N} A_{ij} x_j \qquad x = 1, 2, 3, \ldots, M$

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_M \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{M1} & A_{M2} & \cdots & A_{MN} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_N \end{pmatrix}$$

operating A on $\hat{e}_j$ (w/all components 0 except the j^{th})

$$A\hat{e}_j = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{M1} & A_{M2} & \cdots & A_{MN} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} A_{1j} \\ A_{2j} \\ \vdots \\ \vdots \\ A_{Mj} \end{pmatrix}$$

$$\begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{pmatrix} \cdot \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{pmatrix}$$

$$\begin{aligned} P_{11} &= A_{11}B_{11} + A_{12}B_{21} + A_{13}B_{31} \\ P_{21} &= A_{21}B_{11} + A_{22}B_{21} + A_{23}B_{31} \\ P_{12} &= A_{11}B_{12} + A_{12}B_{22} + A_{13}B_{32} \\ P_{22} &= A_{21}B_{31} + A_{22}B_{22} + A_{23}B_{32} \end{aligned}$$

$A(BC) \equiv (AB)C$

$P = AB$

$Q = BA$

in general $AB \neq BA$

The Null and Identity Matrices

$$AO = O = OA$$

null

$$A + O = O + A = A$$

$$AI = IA = A$$

identity

8.15 Change of Basis and Similarity Transformations

have so far considered vector $\vec{x}$ as geometrical quantity indpt. of basis

we got a basis $\hat{e}_i : i = 1, 2, \ldots, N$

$$\vec{x} = x_1 \hat{e}_1 + x_2 \hat{e}_2 + \ldots + x_N \hat{e}_N$$

its column matrix, is $\vec{x}$ transposed (T):

$$x = (x_1 \quad x_2 \quad \ldots \quad x_N)^T$$

consider how these components change as result of prescribed basis $\hat{e}_i'$, $i = 1, 2, \ldots, N$

related to old basis as: $\hat{e}_j' = \sum_{i=1}^{N} S_{ij} \hat{e}_i$ — S_{ij} is ith component of $\hat{e}_j$ wrt. old basis

any arbitrary $\vec{x}$ vector:

$$\vec{x} = \sum_{i=1}^{N} x_i \hat{e}_i = \sum_{j=1}^{N} x_j' \hat{e}_j' = \sum_{j=1}^{N} x_j' \sum_{i=1}^{N} S_{ij} \hat{e}_i$$

$$x_i = \sum_{j=1}^{N} S_{ij} x_j' \quad \text{or} \quad x = S x'$$

S is transformation matrix

$$\rightarrow \quad S^{-1} x = S^{-1} S x' \quad \rightarrow \quad x' = S^{-1} x$$

* components of $\vec{x}$ transform inversely to the way in which the basis vectors $\hat{e}_i$ themselves transform

transformation law for components of a linear operator under same change of basis

operator eq.^{n} $\vec{y} = \mathcal{A} \vec{x}$ can be written as

$$y = A x \qquad y' = A' x'$$

$$S y' = A S x' \rightarrow y' = S^{-1} A S x'$$

linear operator $\mathcal{A}$ transforms as $A' = S^{-1} A S$

example of similarity transformation

Given square matrix A, may interpret is representing linear operator A in given basis $\hat{e}_i$

can also consider matrix $A' = S^{-1}AS$ for any non-singular matrix S as representing same linear operator A in new basis $\hat{e}'_j$

$$\hat{e}'_j = \sum_i S_{ij} \hat{e}_j$$

Any property of matrix A that represents some property of linear operator A will also be shared by A'

① if $A = 1$, then $A' = 1$

$$A' = S^{-1} 1 S = S^{-1}S = 1$$

② Value of determinant is unchanged:

$$|A'| = |S^{-1}AS| = |S^{-1}||A||S| = |A||S^{-1}||S| = |A||S^{-1}S| = |A|$$

③ characteristic of determinant & eigen values of A' are same from A

$$|A' - \lambda 1| = |S^{-1}AS - \lambda 1| = |S^{-1}(A - \lambda 1)S|$$
$$= |S^{-1}||S||A - \lambda 1| = |A - \lambda 1|$$

④ Value of trace is unchanged

An impt. class of similarity transformations is that for which S is a unitary matrix

$$\longrightarrow \quad A' = S^{-1}AS = S^{\dagger}AS$$

if original basis $\hat{e}_i$ is orthonormal & S is unitary:

$$\langle \hat{e}'_i | \hat{e}'_j \rangle = \langle \sum_k S_{ki} \hat{e}_k | \sum_r S_{rj} \hat{e}_j \rangle$$
$$= \sum_k S^*_{ki} \sum_r S_{rj} \langle \hat{e}_k | \hat{e}_j \rangle = \sum_k S^*_{ki} \sum_r S_{rj} \delta_{kr}$$
$$= \sum_k S^*_{ki} S_{kj} = (S^{\dagger}S)_{ij} = \delta_{ij}$$

new basis is also orthonormal