def^n impt. general examples titles solution methods

in ch. 5, looked @ stationary values of f^n's of a single variable, several variables, & constrained variables

Now bringing about a particular condition for given expression by varying the f^n's on which the expression depends

Concerned w/ general maximisation or minimisation criterion by which the f^n $y(x)$ satisfies the problem

Want to study value of an integral whose integrand has specified form in terms of a certain f^n & its derivatives

Also look @ how value changes when f^n is varied

Goal: Find the f^n making the integral stationary (making value of integral a local max./min.)

22.1 The Euler-Lagrange Equation

Consider: $I = \int_a^b F(y, y', x)\,dx$ ① $y = y(x)$

$y(x)$ is to be chosen to make value of I stationary

↳ functional: $I = I[y(x)]$

Consider making replacement: $y(x) \to y(x) + \alpha\,\eta(x)$ ②

α is small

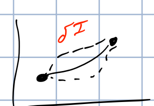
$\eta(x)$ is arbitrary f^n

for value of I to be stationary, need: $\left.\frac{dI}{d\alpha}\right|_{\alpha=0} = 0$ for all $\eta(x)$

subst. ② into ① $\to$ $I(y, \alpha) = \int_a^b F(y + \alpha\eta, y' + \alpha\eta', x)\,dx$

Taylor in α

$= \int_a^b F(y, y', x)\,dx + \underbrace{\int_a^b \left(\frac{\partial F}{\partial y}\alpha\eta, \frac{\partial F}{\partial y'}\alpha\eta'\right) dx}_{\delta I} + O(\alpha^2)$



change in I

$$\delta I = \int_a^b \left(\frac{\partial F}{\partial y}\alpha n + \frac{\partial F}{\partial y'}\alpha n'\right)dx = 0$$

integration by parts $\Rightarrow$

$$\underbrace{\left[n\,\frac{\partial F}{\partial y'}\right]_a^b}_{=0 \text{ b/c } n(a)=n(b)=0} + \int_a^b \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right]\right) n(x)\,dx = 0$$

$$\rightarrow \quad \frac{\partial F}{\partial y} = \frac{d}{dx}\,\frac{\partial F}{\partial y'} \qquad (3)$$

22.2 Special Cases

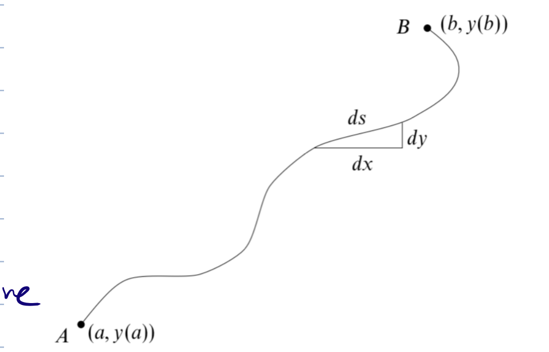


Figure 22.2 An arbitrary path between two fixed points.

F does not contain y explicitly

with this: $\frac{\partial F}{\partial y} = 0 \xrightarrow{(3)} \frac{\partial F}{\partial y'} = \text{constant}$

show shortest curve joining 2 points is a straight line

2 constraints: $(a, y(a))$ + $(b, y(b))$

$$ds = \sqrt{dx^2 + dy^2}\cdot\frac{dx}{dx} = \sqrt{\frac{dx^2}{dx^2} + \frac{dy^2}{dx^2}}\,dx = \sqrt{1+y'^2}\,dx$$

total path length: $L = \int_a^b \underbrace{\sqrt{1+y'^2}}_{\rightarrow F}\,dx$

$$k = \frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}}$$

$$\frac{1}{k} = \frac{\sqrt{1+y'^2}}{y'} = \sqrt{\frac{1}{y'^2} + 1} \qquad \frac{1}{k^2} - 1 = \frac{1-k^2}{k^2} = \frac{1}{y'^2}$$

$$\rightarrow \quad y' = \sqrt{\frac{k^2}{1-k^2}}$$

$$\rightarrow \quad y = \underbrace{\frac{k}{\sqrt{1-k^2}}}_{m}\,x + C$$

$$m = \frac{y(b) - y(a)}{b-a} \quad \leftarrow \quad y = mx + C$$

$y' = m$

$$L = \sqrt{1+m^2}\int_a^b dx = \sqrt{1+m^2}\,(b-a)$$

$$L^2 = (1+m^2)(b-a)^2 = \frac{(b-a)^2 + (y(b)-y(a))^2}{\cancel{(b-a)^2}}\cdot\cancel{(b-a)^2} = (b-a)^2 + (y(b)-y(a))^2$$

F does not contain x explicitly

product rule

$$(3) * y' \rightarrow \frac{d}{dx}\left[y'\,\frac{\partial F}{\partial y'}\right] = y'\underbrace{\frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right]}_{\rightarrow \text{E-L}} + y''\,\frac{\partial F}{\partial y'}$$

$$\rightarrow \quad y'\frac{\partial F}{\partial y} + y''\frac{\partial F}{\partial y'} = \frac{d}{dx}\left[y'\frac{\partial F}{\partial y'}\right]$$

b/c F doesn't depend on x explicitly $\rightarrow$ $\frac{dF}{dx} = \frac{\partial F}{\partial y}\cdot\frac{dy}{dx} + \frac{\partial F}{\partial y'}\cdot\frac{dy'}{dx} = y'F_y + F_{y'}y''$

$$\frac{d}{dx}[F] = \frac{d}{dx}\left[y'\frac{\partial F}{\partial y'}\right]$$

$$\frac{d}{dx}\left[F - y'\frac{\partial F}{\partial y'}\right] = 0$$

$$F - y'\frac{\partial F}{\partial y'} = \text{constant} \qquad (4)$$

22.3 Some Extensions

consider problems w/several dependent and/or indpt. variables or higher-order derivatives of the dependent variable

Several Dependent Variables

$F = F(y_1, y_1', y_2, y_2', \ldots, y_n, y_n', x) \qquad y_i = y_i(x)$

n separate but simultaneous eq^{n}'s

$$\frac{\partial F}{\partial y_i} = \frac{d}{dx}\left(\frac{\partial F}{\partial y_i'}\right) \qquad i = 1, 2, \ldots, n$$

Several Independent Variables

w/n indpt. variables, need to extremise multiple integrals

$$I = \iint \cdots \int F\left(y, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, \ldots, \frac{\partial x}{\partial x_n}, x_1, x_2, \ldots, x_n\right) dx_1 dx_2 \ldots dx_n$$

$y = y(x_1, x_2, \ldots, x_n)$ must satisfy

$$\frac{\partial F}{\partial y} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial F}{\partial y_{x_i}}\right)$$

Higher Order Derivatives

w/ $F = F(y, y', y'', \ldots, y^{(n)}, x)$ do repeated integration by parts

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) - \cdots + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^{(n)}}\right) = 0$$

$y = y' = \cdots = y^{n-1} = 0$ @ endpoints

Variable end points

$$I = \int_a^b F(y, y', x)\, dx$$

demand only lower end-point is fixed, allowing $y(b)$ to be arbitrary

require: $$\left[\eta \frac{\partial F}{\partial y'}\right]_a^b + \int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right] \eta(x)\, dx = 0$$

fixed lower, and $\eta(a) = 0$ for LHS to vanish, $\left.\frac{\partial F}{\partial y'}\right|_b = 0$

if both ends vary, $\frac{\partial F}{\partial y'}$ must vanish @ both ends

general case: lower endpoint fixed, upper end point free to lie on $h(x,y) = 0$

variation in I due to arbitrary constant given to first order:

$$\delta I = \left[\frac{\partial F}{\partial y'} \eta\right]_a^b + \int_a^b \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right) \eta\, dx + F(b)\, \Delta x$$

$\delta I = 0$

$$0 = \left[\frac{\partial F}{\partial y'} \eta\right]_a^b + F(b)\, \Delta x$$

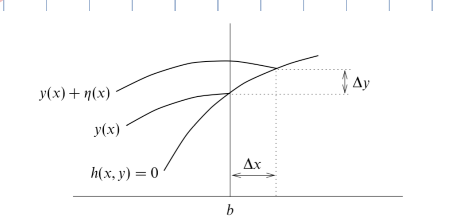


Figure 22.5 Variation of the end-point b along the curve $h(x, y) = 0$.

$\Delta y = \eta(b) + y'(b)\, \Delta x$

b/c $h(x,y) = 0$ @ $x = b$ $\qquad \frac{\partial h}{\partial x} \Delta x + \frac{\partial h}{\partial y} \Delta y = 0$

$$\frac{\partial h}{\partial x} \Delta x + \frac{\partial h}{\partial y}(\eta + y' \Delta x) = \left(\frac{\partial h}{\partial x} + y' \frac{\partial h}{\partial y}\right) \Delta x + \frac{\partial h}{\partial y} \eta = 0$$

⋮ algebra

$$\left(F - y' \frac{\partial F}{\partial y'}\right) \frac{\partial h}{\partial y} - \frac{\partial F}{\partial y'} \frac{\partial h}{\partial x} = 0$$

22.4 Constrained Variation

$g(x, y) =$ constant $\rightarrow f(x,y)$ constrained

$I = \int_a^b F(y, y', x)\, dx$ w/ constraint $J = \int_a^b G(y, y', x)\, dx$ held constant

define a new functional: $K = I + \lambda J = \int_a^b (F + \lambda G)\, dx$

& find unconstrained stationary values

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \lambda \left(\frac{\partial G}{\partial y} - \frac{d}{dx} \frac{\partial G}{\partial y'}\right) = 0$$

multiple constraints J_i: $K = I + \sum_1^n \lambda_i J_i$

22.5 Physical Variational Principles

Fermat & Hamilton