def$^n$ impt. general examples titles solution methods

## 21.1 Seperation of Variables: the general method

suppose we want solution $u(x,y,z,t)$ to some PDE

$$u(x,y,z,t) = X(x)\,Y(y)\,Z(z)\,T(t)$$

separation of variables – solution w/ form said to be separable in $x,y,z,t$

$xyz^2 \sin(bt) \longrightarrow$ completely separable

$xy + zt \longrightarrow$ inseparable

$(x^2+y^2)z\cos(\omega t) \longrightarrow$ separable in $x,y$ & $z$, not $x$ & $y$

require only that $X$ does not depend upon $y,z,t$, same w/ $Y$

look @ wave eq$^n$: $\nabla^2 u(\vec{r}) = \frac{1}{c^2}\frac{\partial^2 u(\vec{r})}{\partial t^2}$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

$$\longrightarrow \frac{\partial^2 X}{\partial x^2} YZT + X\frac{d^2 Y}{dy^2} ZT + XY\frac{d^2 Z}{dz^2} = \frac{1}{c^2} XYZT''$$

$$\xrightarrow{\div\, u = XYZT} \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = \frac{1}{c^2}\cdot\frac{T''}{T}$$

$$\frac{X''}{X} = -l^2 \qquad \frac{Y''}{Y} = -m^2 \qquad \frac{Z''}{Z} = -n^2 \qquad \frac{1}{c^2}\frac{T''}{T} = -\mu^2$$

only for all $x,y,z,t$ if each of the term does not depend upon the corresponding variable but equal to a constant

we get 4 separate ODEs

$$X(x) = A e^{ilx} + B e^{-ilx} \qquad X(x) = A'\cos(lx) + B'\sin(lx)$$
$$Y(y) = C e^{imy} + D e^{-imy} \qquad Y(y) = C'\cos(my) + D'\sin(my)$$
$$Z(z) = E e^{inz} + F e^{-inz} \qquad Z(z) = E'\cos(nz) + F'\sin(nz)$$
$$T(t) = G e^{c\mu t} + H e^{-c\mu t} \qquad T(t) = G'\cos(c\mu t) + H'\sin(c\mu t)$$

take particular sol$^n$'s: $X = e^{ilx}$, $Y = e^{inz}$, $Z = e^{imy}$, $T = e^{-ic\mu t}$

particular sol$^n$: $u(x,y,z,t) = e^{i(lx+my+nz-c\mu t)}$

plane wave of unit amplitude propagating in a direction given by vector w/ compenents $l,m,n$ in Cartesian coords

$$u(x,y,z,t) = e^{i(k_x x + k_y y + k_z z - \omega t)} = e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

Obtain 1-D diffusion eq$^{n}$ $k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$

$$u(x,t) = X(x)\,T(t)$$

$u = XT$

$$\frac{X''}{X} = \frac{T'}{kT}$$

LHS is f$^{n}$ of x
RHS is f$^{n}$ of t

$$\frac{X''}{X} = -\lambda^2 \qquad X'' = -\lambda^2 X \qquad X'' + \lambda^2 X = 0$$

$$\frac{T'}{T} = -\lambda^2 \qquad T' = -\lambda^2 kT \qquad T' + \lambda^2 kT = 0$$

$$\rightarrow \quad X = A\cos(\lambda x) + B\sin(\lambda x)$$
$$Y = C e^{-\lambda^2 kt}$$

$$\xrightarrow{u=XT} \quad u(x,t) = \left(A\cos(\lambda x) + B\sin(\lambda x)\right) e^{-\lambda^2 kt}$$

## 21.2 Superposition of Separated Solutions

hella freedom in values of separation constant $\lambda$

general feature for sol$^{n}$'s in separated form (if PDE has n indpt. variables) $\rightarrow$ n-1 separation constants

take a 2 variable example:

if $u_{\lambda_1}(x,y) = X_{\lambda_1}(x)\,Y_{\lambda_1}(y)$ is a sol$^{n}$ of linear PDE obtained by giving separation constant value $\lambda_1$

superposition:

$$u(x,y) = a_1 X_{\lambda_1}(x)\,Y_{\lambda_1}(y) + a_2 X_{\lambda_2}(x)\,Y_{\lambda_2}(y) + \ldots = \sum_i a_i X_{\lambda_i}(x)\,Y_{\lambda_i}(y)$$

## 21.3 Separation of Variables in Polar Coordinates

polar: $$\vec{\nabla}^2 = \frac{1}{\rho}\cdot\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}$$

cylindrical: $$\vec{\nabla}^2 = \frac{1}{\rho}\cdot\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}$$

spherical: $$\vec{\nabla}^2 = \frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

## Laplace's Equation in Polar Coordinates

Simplest of eq$^{n}$'s containing $\nabla^2$ is Laplaces eq$^{n}$: $\vec{\nabla}^2 u(\vec{r}) = 0$

## Laplace's Eq$^{n}$ in Plane Polars

Suppose we need a solution of Laplace's eq$^{n}$ w/ $\rho = a$

seek sol$^{n}$'s separable in $\rho$ & $\phi$, hope to accomodate BC w/ $\rho = a$

$$u(\rho, \phi) = P(\rho)\,\Phi(\phi)$$

$$\rightarrow \frac{\Phi}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial P}{\partial \rho}\right) + \frac{P}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

divide by $u = P\Phi$
multiply by $\rho^2$

$$\underbrace{\frac{\rho}{P}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial P}{\partial \rho}\right)}_{\text{only dependent on } \rho} + \underbrace{\frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi^2}}_{\text{only dependent on } \phi} = 0$$

$$\frac{\rho}{P}\frac{\partial}{\partial \rho}\left(\rho \frac{dP}{d\rho}\right) = n^2$$

$n$ is complex number
$n^2$ for later convenience

$$\frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi^2} = -n^2$$

consider $n \neq 0$

second eq$^{n}$ has general sol$^{n}$: $\Phi(\phi) = A e^{in\phi} + B e^{-in\phi}$

first eq$^{n}$

$$\frac{\partial}{\partial \rho}\left(\rho \frac{dP}{d\rho}\right) = \rho \frac{d^2P}{d\rho^2} + \frac{dP}{d\rho}$$

$$\frac{\rho}{P}\frac{\partial}{\partial \rho}\left(\rho \frac{dP}{d\rho}\right) = \frac{\rho}{P}\left[\rho \frac{d^2P}{d\rho^2} + \frac{dP}{d\rho}\right] = n^2$$

$$\rightarrow \rho^2 \frac{d^2P}{d\rho^2} + \rho \frac{dP}{d\rho} - n^2 P = 0$$

$$\rho^2 P'' + \rho P' - n^2 P = 0$$

try $\rho = e^{t}$
reduce

$$P(\rho) = C\rho^{n} + D\rho^{-n}$$

if $\Phi$ is single valued & doesn't change when $\phi$ increases by $2\pi$, $n$ must be an integer

Particular sol$^{\underline{n}}$

$$u(\rho,\phi) = [A\cos(n\phi) + B\sin(n\phi)]\cdot[C\rho^n + D\rho^{-n}]$$

A, B, C, D are arbitrary constants, n is any integer

what about $n=0$?

sol$^{\underline{n}}$'s shown as

$\Phi(\phi) = A\phi + B$

$P(\rho) = C\ln(\rho) + D$

for $u$ to be single valued, $A=0$

sol$^{\underline{n}}$: $u(\rho,\phi) = C\ln(\rho) + D$

General Sol$^{\underline{n}}$

$$u(\rho,\phi) = (C_0\ln(\rho) + D_0) + \sum_{n=1}^{\infty}\left[A_n\cos(n\phi) + B_n\sin(n\phi)\cdot(C_n\rho^n + D_n\rho^{-n})\right]$$

$n =$ integer values

Laplace's eq$^{\underline{n}}$ in Cylindrical Coords

if there is no $z$ dependence, can be treated as 2-D plane polars

generally tho...

$$u(\rho,\Phi,z) = P(\rho)\,\Phi(\phi)\,Z(z)$$

$$\frac{1}{\rho}\cdot\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\phi^2} + \frac{\partial^2 u}{\partial z^2}$$

divide by $u = P\Phi Z$

$$\frac{1}{P\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{dP}{d\rho}\right) + \frac{1}{\Phi\rho^2}\frac{d^2\Phi}{d\phi^2} + \underbrace{\frac{1}{Z}\frac{d^2Z}{dz^2}}_{\text{only depends on } z} = 0$$

$k^2 = \frac{1}{Z}\frac{d^2Z}{dz^2}$

sol$^{\underline{n}}$:

$Z(z) = Ee^{-kz} + Fe^{kz}$

multiply by $\rho^2$

$$\underbrace{\frac{\rho}{P}\frac{\partial}{\partial\rho}\left(\rho\frac{dP}{d\rho}\right)}_{\text{only depends on } \rho} + \underbrace{\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}}_{\text{only depends on } \phi} + k^2\rho^2 = 0$$

$\Phi$ eq$^{\underline{n}}$

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = \underbrace{-m^2}_{(}$$

$m \neq 0 \rightarrow \Phi(\phi) = C\cos(m\phi) + D\sin(m\phi)$

$m = 0 \rightarrow \Phi(\phi) = C\phi + D$

$\rho$ eqn

$$\frac{\rho}{P}\frac{\partial}{\partial\rho}\left(\rho\frac{dP}{d\rho}\right)+(-m^2)+k^2\rho^2=0$$

multiply by $P$

$$\rho\frac{\partial}{\partial\rho}\left(\rho\frac{dP}{d\rho}\right)+P\left(k^2\rho^2-m^2\right)=0$$

$$\rho^2\frac{d^2P}{d\rho^2}+\rho\frac{dP}{d\rho}+P\left(k^2\rho^2-m^2\right)=0$$

$$\rho^2P''+\rho P'+(k^2\rho^2-m^2)P=0$$

turn into Bessel's eqn of order m w/ $\mu=k\rho$

sol$^n$: $P(\rho)=A\,J_m(k\rho)+B\,Y_m(k\rho)$ → Ch16

note: $Y_m(k\rho)$ is singular @ $\rho=0$

→ when seeking sol$^n$'s for Laplace's eq$^n$ in cylindrical coords. in region w/ $\rho=0$, need $B=0$

## General Solution

complete separated variable sol$^n$:

$$u(\rho,\phi,z)=\left[A\,J_m(k\rho)+B\,Y_m(k\rho)\right]\cdot\left[C\cos(m\phi)+D\sin(m\phi)\right]\cdot\left[Ee^{-kz}+Fe^{kz}\right]$$

## Laplace's Equation in Spherical Coords

let's try sol$^n$ of form: $u(r,\theta,\phi)=R(r)\,\Theta(\theta)\,\Phi(\phi)$

$$\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}=0$$

divide by $u=R\Theta\Phi$, multiply by $r^2$

R

$$\underbrace{\frac{1}{R}\cdot\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)}_{\text{depends only on } r}+\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{1}{\Phi\sin^2\theta}\frac{d^2\Phi}{d\theta}=0$$

first term

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=\lambda\;\rightarrow\;\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=\lambda R$$

$$\rightarrow r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}-\lambda R=0$$

why?

how:

$$\to \frac{d^2S}{dt^2} + \frac{dS}{dt} - \lambda S = 0$$

$sol^n$: $S(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$

$$R(r) = Ar^{\lambda_1} + Br^{\lambda_2}$$

where ... $\lambda_1 + \lambda_2 = -1$ & $\lambda_1 \cdot \lambda_2 = -\lambda$

can take $\lambda_1$ & $\lambda_2$ as $\ell$ & $-(\ell+1)$

$\lambda$ has form $\ell(\ell+1)$

can rewrite $u(r,\theta,\phi)$ & PDE

$$u(r,\theta,\phi) = R\,\Theta\,\Phi$$

$$= \left[Ar^{\ell} + Br^{-(\ell+1)}\right]\Theta(\theta)\,\Phi(\phi)$$

$$\underbrace{\frac{1}{R}\cdot\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)}_{\lambda} + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Theta\sin^2\theta}\frac{d^2\Theta}{d\theta} = 0$$

$$\to \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Theta\sin^2\theta}\frac{d^2\Theta}{d\theta} = -\lambda = -\ell(\ell+1)$$

multiply by $\sin^2\theta$

$$\to \left[\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \ell(\ell+1)\sin^2\theta\right] + \underbrace{\frac{1}{\Theta\sin^2\theta}\frac{d^2\Theta}{d\theta}}_{-m^2} = 0$$

$\Phi$ $eq^n$

repeat as in cylindrical $sol^n$

$sol^n$: $\Phi(\phi) = C\cos(m\phi) + D\sin(m\phi)$, for $m \neq 0$

if $m = 0$, $\Phi(\phi) = C\phi + D$

rewrite PDE

$$\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \ell(\ell+1)\sin^2\theta = m^2$$

$\Theta$ $eq^n$

change variables from $\theta$ to $\mu$

$\mu = \cos\theta \qquad \frac{d\mu}{d\theta} = -\sin\theta$

$$\frac{d}{d\theta} = -\sin\theta\cdot\frac{d}{d\mu} \to \frac{d}{d\theta} = -\sqrt{1-\cos^2\theta}\,\frac{d}{d\mu} = -\sqrt{1-\mu^2}\,\frac{d}{d\mu}$$

$\Theta(\theta) = M(\mu)$

$\frac{d}{d\theta} \to -\sqrt{1-\mu^2}\,\frac{d}{d\mu}$

$\sin\theta = \sqrt{1-\mu^2}$

$\sin^2\theta \to 1-\mu^2$

rewrite PDE

$$\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + l(l+1)\sin^2\theta = m^2$$

$$\to \frac{\sqrt{1-\mu^2}}{M}\cdot -\sqrt{1-\mu^2}\,\frac{d}{d\mu}\left(\sqrt{1-\mu^2}\cdot -\sqrt{1-\mu^2}\,\frac{dM}{d\mu}\right) + l(l+1)(1-\mu^2) = m^2$$

$$\to \frac{1-\mu^2}{M}\frac{d}{d\mu}\left((1-\mu^2)\frac{dM}{d\mu}\right) + l(l+1)(1-\mu^2) - m^2 = 0$$

multiply $\frac{M}{1-\mu^2}$ $\to$ $$\frac{d}{d\mu}\left((1-\mu^2)\frac{dM}{d\mu}\right) + \left[l(l+1) - \frac{m^2}{1-\mu^2}\right]M = 0$$

associated Legendre's eq$^n$

sol$^n$'s for $P_l^m(\mu)$ & $Q_l^m(\mu)$

for $m=0$, simplifies to Legendre's eq$^n$

$$M(\mu) = E\,P_l(\mu) + F\,Q_l(\mu)$$

Sol$^n$'s for general $m$ found in Section 18.2

$$P_l^m(\mu) = (1-\mu^2)^{|m|/2}\frac{d^{|m|}}{d\mu^{|m|}}\left[P_l(\mu)\right]$$

$$Q_l^m(\mu) = (1-\mu^2)^{|m|/2}\frac{d^{|m|}}{d\mu^{|m|}}\left[Q_l(\mu)\right]$$

$$M(\mu) = E\,P_l^m(\mu) + F\,Q_l^m(\mu)$$

$m$ must be integer $\quad 0 \le |m| \le l$

if we require sol$^n$'s to Laplace's eq$^n$ are finite when $\mu = \cos\theta = \pm 1$

$\to$ must have $F = 0$ b/c $Q_l^m(\mu)$ diverges @ $\mu = \pm 1$

must have $l$ as integer $\ge 0$

General sol$^{\underline{n}}$

$$u(r,\theta,\phi) = R(r)\,\Phi(\phi)\,\Theta(\theta)$$

$$= \left[Ar^{l} + Br^{-(l+1)}\right] \cdot \left[C\cos(m\phi) + D\sin(m\phi)\right] \cdot \left[E P_l^m \cos(\theta) + F Q_l^m \cos(\theta)\right]$$

where the three are connected only thru integer parameters $l$ & $m$ : $0 \leq |m| \leq l$

if sol$^{\underline{n}}$ is required to be finite on the polar axis then $F = 0$ $\forall\, m, l$