def^n impt. general examples titles solution methods

PDE - equation relating an unknown f^n of 2 or more variables to its partial derivatives

look @ linear PDEs, primarily second order

20.1 Important Partial Differential Equations

actual variables used to specify position vector $\vec{r}$ depends on coordinate system in use

Wave Equation $$\vec{\nabla}^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

describes f^n of position & time the displacement from equilibrium of a vibrating string or membrane or vibrating liquid, solid, or gas

Diffusion Equation $$\kappa \vec{\nabla}^2 u = \frac{\partial u}{\partial t}$$

describes temp. u in region containing no heat sources or sinks

Laplace's Equation $$\vec{\nabla}^2 u = 0$$

Poissons Equation $$\vec{\nabla}^2 u = \rho(\vec{r})$$

Schrodinger's Equation $$-\frac{\hbar}{2m}\vec{\nabla}^2 u + V(\vec{r})u = i\hbar\frac{\partial u}{\partial t}$$

20.2 General Form of Solution

suppose we have set of f^n's involving 2 indpt. variables x, y

consider type of f^n $u_i(x,y)$ s.t. u_i can be written as a f^n of a single variable p (itself a f^n of x & y)

consider the 3 f^{n}s: $p = x^2 + 2y$

$$u_1(x,y) = x^4 + 4(x^2y + y^2 + 1) = p^2 + 4 = f_1(p)$$
$$u_2(x,y) = \sin(x^2)\cos(2y) + \cos(x^2)\sin(2y) = \sin(p) = f_2(p)$$
$$u_3(x,y) = \frac{x^2+2y+2}{3x^2+6y+5} = \frac{p+2}{3p+5} =$$

$$\frac{\partial}{\partial x}[u_i] = \frac{d f_i(p)}{dp}\frac{\partial p}{\partial x} = 2x f_i'$$

$$\frac{\partial}{\partial y}[u_i] = \frac{d f_i(p)}{dp}\frac{\partial p}{\partial y} = 2 f_i'$$

b/c $\frac{\partial u_i}{\partial x} = \frac{\partial u_i}{\partial y} \rightarrow \frac{\partial p}{\partial y}\frac{\partial u_i}{\partial x} = \frac{\partial p}{\partial x}\frac{\partial u_i}{\partial y}$ $\qquad \frac{\partial p}{\partial x} = 2x$, $\frac{\partial p}{\partial y} = 2$

$$\rightarrow \quad 2 \cdot \frac{\partial u_i}{\partial x} = 2x \frac{\partial u_i}{\partial y}$$

$$\frac{\partial u_i}{\partial x} = x \frac{\partial u_i}{\partial y}$$

20.3 General and Particular Solutions

not generally true that an nth order PDE can always be considered as resulting from elimination of n arbitrary fn's from its solution

given specific PDEs, can try to solve them by seeking combinations of variables in terms of which the sol^n's may be expressed as arbitrary fn's

exact form must be determined by some set of boundary conditions

if PDE contains 2 indpt. variables x & y, complete determination of solution the boundary conditions will take form equivalent to specifying $u(x,y)$ along continuum of points in the x-y plane

First order equations

most general form: $\underbrace{A(x,y)}\frac{\partial u}{\partial x} + \underbrace{B(x,y)}\frac{\partial u}{\partial y} + \underbrace{C(x,y)}u = \underbrace{R(x,y)}$ → given fn's

Find general sol^n of $\quad x\frac{\partial u}{\partial x} + 3u = x^2$

$$\frac{\partial u}{\partial x} + 3\frac{1}{x}u = x \qquad \text{try} \cdot x^3$$

$$x^3\frac{\partial u}{\partial x} + 3\frac{x^3}{x}u = x^3 \cdot x$$

$$x^3\frac{\partial u}{\partial x} + 3x^2u = x^4$$

$$\frac{d}{dx}\left[x^3 u\right] = x^4$$

$$x^3 u = \frac{1}{5}x^5 + f(y)$$

$$u(x,y) = \frac{1}{5}x^2 + \frac{1}{x^3}f(y)$$

when PDE contains partial derivatives wrt both indpt variables, things get peepoo

let's consider case $R(x,y) = C(x,y) = 0$

$u(x,y) = f(p)$ $\qquad \frac{\partial u}{\partial x} = \frac{\partial f(p)}{\partial p} \cdot \frac{\partial p}{\partial x}$ & $\frac{\partial u}{\partial y} = \frac{\partial f(p)}{\partial p} \cdot \frac{\partial p}{\partial y}$

substituting into general case we find: $\left[A(x,y)\frac{\partial p}{\partial x} + B(x,y)\frac{\partial p}{\partial y}\right] \cdot \frac{\partial f(p)}{\partial p} = 0$

$$A(x,y)\frac{\partial p}{\partial x} + B(x,y)\frac{\partial p}{\partial y} = 0$$

consider p is constant

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy$$

if $\frac{dx}{A(x,y)} = \frac{dy}{B(x,y)}$, can say are the same

For $x\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} = 0$, ① find sol^n taking value $2y+1$ on line $x=1$
② find sol^n w/ value 4 @ point $(1,1)$

$A(x,y) = x$ & $B(x,y) = -2y$

$$\frac{dx}{x} = \frac{dy}{-2y} \qquad \ln(x) = \ln(y^{-1/2}) + C$$

$$x = y^{-1/2} + C$$
$$x^2 = y^{-1} \longrightarrow p = x^2 y \qquad u(x,y) = f(x^2y)$$

① $u(x,y) = 2(x^2y) + 1 = 2x^2y + 1$

② $u(x,y) = x^2y + 3$ or $4x^2y$ or 4

in order to find a sol^n of form $u(x,y) = f(p)$, require original PDE contains no terms in u, but only terms containing its partial derivatives

if a term u is present $(c(x) \neq 0)$, can't simply divide out dependence on $f(p)$

look @ $u(x,y) = h(x,y)\, f(p)$

Find general sol^n of $x\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} - 2u = 0$

need to have form $u(x,y) = h(x,y)\, f(p)$ s.t.

$$\frac{\partial u}{\partial x} = \frac{\partial h}{\partial x}f(p) + h\frac{\partial f(p)}{\partial p}\frac{\partial p}{\partial x} \quad \& \quad \frac{\partial u}{\partial y} = \frac{\partial h}{\partial y}f(p) + h\frac{\partial f(p)}{\partial p}\frac{\partial p}{\partial y}$$

substituting in

$$x\left[\frac{\partial h}{\partial x}f(p) + h\frac{\partial f(p)}{\partial p}\frac{\partial p}{\partial x}\right] + 2\left[\frac{\partial h}{\partial y}f(p) + h\frac{\partial f(p)}{\partial p}\frac{\partial p}{\partial y}\right] - 2hf = 0$$

$$x\frac{\partial h}{\partial x}f(p) + xh\frac{\partial f(p)}{\partial p}\frac{\partial p}{\partial x} + 2\frac{\partial h}{\partial y}f(p) + 2h\frac{\partial f(p)}{\partial p}\frac{\partial p}{\partial y} - 2hf(p) = 0$$

$$f(p)\left[x\frac{\partial h}{\partial x} + 2\frac{\partial h}{\partial y} - 2h\right] + \frac{\partial f(p)}{\partial p}\left[xh\frac{\partial p}{\partial x} + 2h\frac{\partial p}{\partial y}\right] = 0$$

$$f(p)\underbrace{\left[x\frac{\partial h}{\partial x} + 2\frac{\partial h}{\partial y} - 2h\right]}_{\text{Same as original PDE w/ } h=u} + \frac{\partial f(p)}{\partial p}\, h\left[x\frac{\partial p}{\partial x} + 2\frac{\partial p}{\partial y}\right] = 0$$

if h is any sol^n of PDE, term will vanish

$$\underbrace{x}_{A(x,y)}\frac{\partial p}{\partial x} + \underbrace{2}_{B(x,y)}\frac{\partial p}{\partial y} = 0 \rightarrow \frac{dx}{x} = \frac{dy}{2}$$

$$\ln(x) = \tfrac{1}{2}y$$

$$x = ce^{\frac{1}{2}y} \quad c \text{ constant of integration}$$

$$p = xe^{-\frac{1}{2}y}$$

$$u(x,y) = h(x,y)\cdot f(xe^{-\frac{1}{2}y})$$

where $f(p)$ is any arbitrary f^n of p & $h(x,y)$ is any sol^n of PDE

Inhomogeneous equations and Problems

if $u(x,y)$ is a sol^n then so is a multiple of $u(x,y)$ or any linear sum of seperate sol^ns $u_1(x,y) + u_2(x,y)$

equation said to be homogeneous if $u(x,y)$ is a sol^n implies $\lambda u(x,y)$ is a sol^n (constant)

also homogeneous if (in addition) if boundary conditions are satisfied by $u(x,y)$, so is $\lambda u(x,y)$

→ like particular & complementary sol^n for ODEs

general sol^n for inhomogeneous problem can be written as sum of any particular sol^n & general sol^n

Find general Solution of $y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 3x$

Consider w/o regard for boundary conditions, let's look @ $y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 0$

$$A(x,y) = y \quad B(x,y) = -x \quad \frac{dx}{A(x,y)} = \frac{dy}{B(x,y)} \rightarrow \frac{dx}{y} = \frac{dy}{-x}$$

$\longrightarrow \quad -x\,dx = y\,dy \rightarrow -\frac{1}{2}x^2 = \frac{1}{2}y^2 \rightarrow \frac{1}{2}(y^2+x^2) = C$

$p = 2C = y^2+x^2$

find general sol^{n} : $u(x,y) = f(y^2+x^2)$ for arbitrary f^{n} f

by inspection: $u(x,y) = -3y$ as particular sol^{n}

$\longrightarrow u(x,y) = f(y^2+x^2) - 3y$

for $u(x,0) = x^2 = f(x^2)$

$u(x,y) = x^2+y^2-3y$

for $u(1,y) = 2$ @ $(1,0)$

$u(1,0) = f(1) = 2$

$u(x,y) = 2x^2+2y^2-3y+g(x^2+y^2)$

Second Order Equations

most general second order PDE has form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = R(x,y)$$

3 classes
1. hyperbolic if $B^2 > 4AC$
2. parabolic if $B^2 = 4AC$
3. elliptic if $B^2 < 4AC$

hard stuff, let's deal w/only complementary $\rightarrow R(x,y) = 0$

require $D = E = F = 0$

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\partial y} + C\frac{\partial^2 u}{\partial y^2} = 0$$

$\rightarrow$ of form of 1-D wave eq^{n}, & 2-D Laplace eq^{n}

b/c terms involve 2 differentiations (assuming solution $u(x,y) = f(p)$) may be able to obtain common factor $\frac{d^2 f(p)}{dp^2}$

consider $\frac{\partial u}{\partial x} = \frac{df(p)}{dp}\frac{\partial p}{\partial x}$ won't lead to single term on RHS containing f as $\frac{d^2 f(p)}{dp^2}$

$\longrightarrow$ p must be linear f^{n} of x & linear to y assume form $u(x,y) = f(ax + by)$

$$\frac{\partial u}{\partial x} = a \frac{df(p)}{dp} \qquad \frac{\partial u}{\partial y} = b \frac{df(p)}{dp}$$

$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{d^2 f(p)}{dp^2} \qquad \frac{\partial^2 u}{\partial x \partial y} = ab \frac{d^2 f(p)}{dp} \qquad \frac{\partial^2 u}{\partial y^2} = b^2 \frac{d^2 f(p)}{dp^2}$$

$$\longrightarrow (Aa^2 + Bab + Cb^2) \frac{d^2 f(p)}{dp^2} = 0$$

$$Aa^2 + Bab + Cb^2 = 0$$