def^n impt. general examples titles solution methods

we've developed methods for solving some eq^n's where the coefficients were not constant, but f^n's of x

→ able to write sol^n's in terms of elementary f^n's

→ usually not write in format to solve

16.1 Second-Order Linear ODEs

any homogeneous second order linear ODE can be written in form

$$y'' + p(x)\,y' + q(x)\,y = 0$$

$$y'' = -(p(x)y' + q(x)y)$$

from chapter 15, we know sol^n written in form

$$y(x) = \underbrace{c_1 \underbrace{y_1(x)} + c_2 \underbrace{y_2(x)}}_{\text{linearly indpt.}}$$

to be linearly indpt., y_2 is not a multiple of y_1

$$c_1 y_1(x) + c_2 y_2(x) = 0 \quad \text{iff} \quad c_1 = c_2 = 0$$

verified w/ Wronskian f^n

$$W(x) = \ldots = y_1 y_2' - y_2 y_1'$$

if $W(x) \neq 0$ in on interval, y_2 & y_1 are linearly indpt.

$$W' = y_1 y_2'' + y_1' y_2' - y_2 y_1'' - y_2' y_1' = y_1 y_2'' - y_2 y_1''$$

$$= -y_1 (p y_2' + q y_2) + (p y_1' + q y_1)\, y_2 = -p(y_1 y_2' - y_2 y_1') = -pW$$

integrate

$$W(x) = C\, e^{-\int_0^x p(u)\,du}$$

Ordinary and Singular Points of an ODE

$y(x)$ can be complex, generalize a $y(z)$

consider

$$y'' + p(z)\,y + q(z) = 0 \quad ④$$

$$\frac{dy}{dz} = y'$$

if @ some point $z = z_0$ & functions $p(z)$ & $q(z)$ are finite & able to express in complex power series

$$\longrightarrow \quad p(z) = \sum_{n=0}^{\infty} p_n (z-z_0)^n \qquad q(z) = \sum_{n=0}^{\infty} q_n (z-z_0)^n$$

if $p(z)$ or $q(z)$ or both diverge @ $z = z_0$ $\longrightarrow$ singular point of the ODE

$p(z)$ & $q(z)$ are analytic @ $z = z_0$ $\longrightarrow$ ordinary point

16.2 Series solutions about an ordinary point

if $z = z_0$ is an ordinary point, can be shown every sol^n $y(z)$ is also analytic @ $z = z_0$

take z_0 as origin

b/c every sol^n is analytic $y(z)$ can take form

$$y(z) = \sum_{n=0}^{\infty} a_n z^n \quad ①$$

$|z| < R$

R is radius of convergence
@ R, series may or may not converge

always possible to obtain 2 indpt sol^n b/c every sol^n is analytic @ an ordinary point

$$y' = \sum_{n=0}^{\infty} n a_n z^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n \quad ②$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n z^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} z^n \quad ③$$

first 2 terms are 0
shift summation index to have z^n

By substituting eq^n ① ② ③ into eq^n ④ & having coefficients of each power of z to sum to 0

$\longrightarrow$ a recurrance relation, expressing each a_n in terms of the previous a_r $(0 \le r \le n-1)$

16.5 Polynomial Solutions

have seen evaluation of successive terms of a series sol^n to an ODE carried out by recurrence relation

a_n depends on n, values of a_r $(r<n)$, & parameters of eq^n

may happen for some value $n = N+1$, computed value a_N is zero & all higher a_r also vanish

corresponding sol^n of indicial eq^n σ is a positive or zero, left w/ polynomial of degree $N' = N + \sigma$ as sol^n of ODE

$$y(z) = \sum_{n=0}^{N} a_n z^{n+\sigma} \quad (5)$$

termination of potentially infinite series after finite # of terms is super impt in theoretical physics

simpler method of obtaining finite polynomial sol^{n}s is assume form of eq^n (5) w/ $a_N \neq 0$

now we start by considering coefficient of the highest power z^N