def^n impt.

general

examples

titles

solution methods

linear ODE of general order n has form

$$a_n(x)\frac{\partial^n y}{\partial x^n}+a_{n-1}(x)\frac{\partial^{n-1} y}{\partial x^{n-1}}+\dots+a_1(x)\frac{\partial y}{\partial x}+a_0(x)\,y=f(x)$$

if $f(x)=0$, the eq^n is homogeneous

to find general solution, should first find solution of complementary equation ($f(x)=0$)

$$a_n(x)\frac{\partial^n y}{\partial x^n}+a_{n-1}(x)\frac{\partial^{n-1} y}{\partial x^{n-1}}+\dots+a_1(x)\frac{\partial y}{\partial x}+a_0(x)\,y=0$$

to solve, must find n linearly indpt. functions satisfying it $\rightarrow$ ends up being roots

if the solutions are $y_1, y_2, y_3, \dots, y_n$, general solution is given by the linear superposition / complementary function

$$y_c(x)=c_1y_1(x)+c_2y_2(x)+\dots+c_ny_n(x)$$

c_m - arbitrary constants

for n functions to be linearly indpt over interval, w/constants $c_1, c_2, \dots, c_s$, it must be

$$c_1y_1(x)+c_2y_2(x)+\dots+c_ny_n(x)\neq 0$$

except for $c_1=c_2=\dots=c_n=0$

an equivalent statement:

$$c_1y_1(x)+c_2y_2(x)+\dots+c_ny_n(x)\neq 0$$
$$c_1y_1'(x)+c_2y_2'(x)+\dots+c_ny_n'(x)\neq 0$$
$$\vdots$$
$$c_1y_1^{n-1}(x)+c_2y_2^{n-1}(x)+\dots+c_ny_n^{n-1}(x)\neq 0$$

Wronskian f^n

if OG eq^n has $f(x)=0$, complementary f^n $y_c(x)$ is the general solution

if $f(x)\neq 0$, need a f^n $y_p(x)$, any f^n satisfying OG eq^n directly being linearly indpt. of $y_c(x)$, the particular integral

$$y(x)=y_c(x)+y_p(x)$$

15.1 Linear Equations w/ Constant Coefficients

if we have constants instead of f^n's of x for a_m from OG eq^n

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1 \frac{dy}{dx} + a_0 y = f(x)$$

again, if $f(x) = 0$, complementary f^n is the general solution

Finding the complementary function $y_c(x)$

complementary f^n must satisfy

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1 \frac{dy}{dx} + a_0 y = 0$$

w/ n arbitrary constants

Standard method to find solution: $y = Ae^{\lambda x}$

$$a_n A e^{\lambda x} \cdot \lambda^n + a_{n-1} A \cdot e^{\lambda x} \cdot \lambda^{n-1} + \dots + a_1 A e^{\lambda x} \lambda + a_0 A e^{\lambda x} = 0$$

$$\underbrace{Ae^{\lambda x}} \left(a_n \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_1 \lambda + a_0 \right) = 0$$

will never be zero, safe to divide out

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_1 \lambda + a_0 = 0$$

auxiliary eq^n!

in general, auxiliary eq^n has n roots : $\lambda_1, \lambda_2, \dots, \lambda_n$

3 cases

① All roots are real & distinct

complementary f^n : $y_c(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + \dots + C_n e^{\lambda_n x}$

② Some roots are complex, still of same form except exponents are now complex

if a complex # $\alpha + i\beta$ is a root, its conjugate is also a root $\alpha - i\beta$

$$c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x} = e^{\alpha x}\left(d_1 \cos(\beta x) + d_2 \sin(\beta x)\right)$$

$$= Ae^{\alpha x} \left\{ \begin{matrix} \sin \\ \cos \end{matrix} \right\} (\beta x + \phi)$$

③ Some roots repeated

have not found n linearly indpt. solutions

if λ_1 occurs k times, must find $k-1$ linearly indpt. solutions

$$x e^{\lambda_1 x}, \quad x^2 e^{\lambda_1 x}, \quad \dots, \quad x^{k-1} e^{\lambda_1 x}$$

$$\rightarrow \quad y_c(x) = (c_1 + c_2 t \dots + c_k x^{k-1}) e^{\lambda_1 x} + C_{k+1} e^{\lambda_{k+1} x} + C_{k+2} e^{\lambda_{k+2} x} + \dots + C_n e^{\lambda_n x}$$

repeat if more than 1 root repeats

$$y_c(x) = (C_1 + C_2 x + \dots + C_k x^{k-1}) e^{\lambda_1 x} + (C_{k+1} + C_{k+2} x + \dots + C_{k+l} x^{l-1}) e^{\lambda_2 x} + C_{k+l+1} e^{\lambda_{k+l+1} x} + C_{k+l+2} e^{\lambda_{k+l+2} x} + \dots \; C_n e^{\lambda_n x}$$

Find complementary f^n of $\quad \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x$

$$y = Ae^{\lambda x}$$

$$\frac{d^2}{dx^2}(Ae^{\lambda x}) - 2\frac{d}{dx}(Ae^{\lambda x}) + Ae^{\lambda x} = 0$$

$$\frac{d}{dx}(A\lambda e^{\lambda x}) - 2Ae^{\lambda x}\lambda + Ae^{\lambda x} = 0$$

$$Ae^{\lambda x}\lambda^2 - 2Ae^{\lambda x}\lambda + Ae^{\lambda x} = 0$$

$$Ae^{\lambda x}(\lambda^2 - 2 + 1) = 0$$

$$\lambda^2 - 2 + 1 = 0$$

$$(\lambda - 1)^2 = 0$$

$\lambda = 1$

twice $\rightarrow k = 2$

$$y_c(x) = (c_1 + c_2 t \dots + c_k x^{k-1}) e^{\lambda_1 x} + C_{k+1} e^{\lambda_{k+1} x} + C_{k+2} e^{\lambda_{k+2} x} + \dots + C_n e^{\lambda_n x}$$

$$y_c(x) = (C_1 + \dots + C_k x^{k-1}) e^{\lambda_1 x}$$

$$y_c(x) = (C_1 + C_2 x^{2-1}) e^{\lambda_1 x}$$

$$y_c(x) = (C_1 + C_2 x) e^x$$

Solving method:

① Set RHS of ODE to 0
② Subst. $y = Ae^{\lambda x}$
③ Divide & obtain n-th order polynomial in λ
④ Solve auxiliary eq^n to find n roots
⑤ Use one of 3 cases

Finding the particular integral

no general method for finding particular integral $y_p(x)$

for linear ODEs w/constant coefficients & a simple RHS, $y_p(x)$ can often be found by inspection or by assuming a parameterised form similar to $f(x)$

method of undetermined coefficients

if $f(x)$ only contains polynomial, exponential, sine or cosine terms.

then by assuming trial function for $y_p(x)$ of similar form
(but one which contains a # of undetermined parameters & subbing into eq^n $a_n \frac{d^n y}{dx^n} \ldots + a_0 y = 0$)

parameters can be found & $y_p(x)$ deduced

Standard trial f^n's:

i) if $f(x) = ae^{rx}$, then try

$$y_p(x) = be^{rx}$$

ii) if $f(x) = a_1 \sin(rx) + a_2 \cos(rx)$ (a_1 or a_2 may be zero), then try

$$y_p(x) = b_1 \sin(rx) + b_2 \cos(rx)$$

iii) if $f(x) = a_0 + a_1 x + \ldots + a_N x^N$ (some a_m may be zero), then try

$$y_p(x) = b_0 + b_1 x + \ldots + b_N x^N$$

iv) if $f(x)$ is the sum or product of any of the above then try $y_p(x)$ as the sum or product of the corresponding indv. trial f^n's

this method fails if any term in assumed trial f^n is also complementary f^n $y_c(x)$

if this is the case, trial f^n should be multiplied by smallest integer power of x s.t. it will then contain no term that already appears in $y_c(x)$

can be subst. into eq^n $a_n \frac{d^n y}{dx^n} + \ldots + a_0 y = 0$ to find coefficients

3 other methods:
1. Green's f^n's
2. variation of parameters
3. change in the dependent variable using knowledge of $y_c(x)$

Find a particular integral of the equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x$$

try $y_p(x) = be^x$

b/c complementary $y_c(x) = (c_1 + c_2x)e^x$, e^x already in it

let's try multiplying by x s.t. it isn't in $y_c(x)$

$y_p(x) = bxe^x$, xe^x already in $y_c(x)$, try again

$\longrightarrow$ $y_p(x) = b\,x^2e^x$

$$\frac{dy_p}{dx} = b\cdot\left(2xe^x + x^2e^x\right) = be^x\left(2x + x^2\right)$$

$$\frac{d^2y_p}{dx^2} = b\left(2\frac{d}{dx}(xe^x) + \frac{d}{dx}(x^2e^x)\right)$$

$$= b\left(2(e^x + xe^x) + (2xe^x + x^2e^x)\right)$$

$$= b\left(2e^x(1+x) + e^x(2x + x^2)\right)$$

$$= be^x\left(2 + 2x + 2x + x^2\right)$$

$$= be^x\left(x^2 + 4x + 2\right)$$

$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x$

$$b\cancel{e^x}(x^2 + 4x + 2) - 2\cdot b\cancel{e^x}(2x + x^2) + x^2\cancel{e^x} = \cancel{e^x}$$

$$b(x^2 + 4x + 2) - 2b(2x + x^2) + x^2 = 1$$

$$bx^2 + 4bx + 2b - 4bx - 2bx^2 + x^2 = 1$$

$$bx^2 + 4bx + 2b - 4bx - 2bx^2 = 1 - x^2$$

$$b(x^2 \cancel{+4x} + 2 \cancel{-4x} - 2x^2) = 1 - x^2$$

$$b(2 - x^2) = 1 - x^2$$

$$b = \frac{1 - x^2}{2 - x^2} \qquad \text{for } x = 0, \quad b = 1/2$$

$y_p(x) = b\,x^2e^x \longrightarrow y_p(x) = \frac{1}{2}x^2e^x$

Constructing the General Solution $y_c(x) + y_p(x)$

full solution to ODE found by adding complementary f^n & any particular integral

Solve $$\frac{d^2y}{dx^2} + 4y = x^2 \sin(2x)$$

Set RHS to zero $\quad \frac{d^2y}{dx^2} + 4y = 0$

$y = Ae^{\lambda x} \quad (\lambda^2 + 4)Ae^{\lambda x} = 0$

$\lambda^2 + 4 = 0$

$\lambda = \pm 2i$

$y_c(x) = c_1 e^{2ix} + c_2 e^{-2ix} = d_1 \cos(2x) + d_2 \sin(2x)$

now $y_p(x)$. good first guess for trial f^n

$(ax^2 + bx + c)\underbrace{\sin(2x)} + (dx^2 + ex + f)\underbrace{\cos(2x)}$

already appear in $y_c(x)$

try multiplying w/ lowest order x not in $y_c(x)$

$(ax^3 + bx^2 + cx)\sin(2x) + (dx^3 + ex^2 + fx)\cos(2x)$

plug in & math

$\longrightarrow y_p(x) = -\frac{x^3}{12}\cos(2x) + \frac{x^3}{16}\sin(2x) + \frac{x}{32}\cos(2x)$

$y_g(x) = y_c(x) + y_p(x) = d_1 \cos(2x) + d_2 \sin(2x) - \frac{x^3}{12}\cos(2x) + \frac{x^3}{16}\sin(2x) + \frac{x}{32}\cos(2x)$