Differential equations – group of equations that contain derivatives

Ordinary Differential Equations (ODEs) contain only ordinary derivatives – no partials

order of an ODE : highest derivative

degree of an ODE : power to which highest-order derivative is raised

$$\frac{d^3y}{dx^3} + x\left(\frac{dx}{dx}\right)^{3/2} + x^2y = 0$$

order : 3rd
degree : 2nd

general solution is most general function satisfying equation

contain n arbitrary constants of integration

particular solution → when boundary conditions have been applied & constants found

singular solution → no arbitrary constants & can't be found from general

14.1 General Form of Solution

① a group of functions w/ n parameters satisfies n-th order ODE in general

→ $y = f(x, a_1, a_2, \ldots, a_n)$

② general solution of an n-th order ODE contains n albitrary parameters

14.2 First-degree first-order equations

first degree, first order ODEs have only dy/dx

can be written in 2 forms : ① $\frac{dy}{dx} = F(x,y) = \frac{-A(x,y)}{B(x,y)}$ ② $A(x,y)\,dx + B(x,y)\,dy = 0$

Separable Variable Eqⁿ's

separable variable eqⁿ is one can be written in form: $\frac{dy}{dx} = f(x)g(y)$

where $f(x)$ & $g(y)$ functions of x & y, even if only a constant

rearranging...

$$\int \frac{dy}{g(y)} = \int f(x)\,dx$$

terms depending on x in one side, y on other → separated can be rearranged to fit this w/factorization

Solve $\frac{dy}{dx} = x + xy$

$$\frac{dy}{dx} = x(1+y) \to \frac{dy}{1+y} = x\,dx \to \int \frac{dy}{1+y} = \int x\,dx$$

$$\to \ln(1+y) = \frac{1}{2}x^2 + c \to 1+y = Ae^{\frac{1}{2}x^2} \qquad A = e^c$$

Solution Method:
1. Factorize to make it separable
2. Rearrange
3. Integrate (remember constant)

Exact Equations

an exact ODE is of form: $A(x,y)\,\partial x + B(x,y)\,\partial y = 0$ & $\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$

this is an exact differential: $A\,dx + B\,dy = dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy$

from which we get $A(x,y) = \frac{\partial U}{\partial x}$ $\qquad B(x,y) = \frac{\partial U}{\partial y}$

We assume continuous & differential top f^n s.t.

$$\frac{\partial^2 U}{\partial x \partial y} = \frac{\partial^2 U}{\partial y \partial x} \rightsquigarrow \frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$$

if this holds, can be written as level curve

$U(x,y) = c$

$$\int A(x,y)\,dx + g(y) \qquad \int B(x,y)\,dy + h(x)$$

Find constant $\frac{\partial}{\partial y}[U(x,y)]$ $\qquad \frac{\partial}{\partial x}[U(x,y)]$

Solve for $g'(y)$ $\qquad h'(x)$

Replace & rearrange

look @ PH13 review notes for detailed steps from YT

Solve $\quad x\frac{dy}{dx} + 3x + y = 0$

$$x\,dy + (3x+y)\,dx = 0$$

$$\frac{\partial A}{\partial y} = 1 \qquad \frac{\partial B}{\partial x} = 1 \qquad \checkmark \text{exact!}$$

$$A(x,y) = 3x+y = \frac{\partial F}{\partial x}$$
$$B(x,y) = x = \frac{\partial F}{\partial y}$$

① $\int A(x,y)\,dx = \int (3x+y)\,dx$

$F(x,y) = \frac{3}{2}x^2 + yx + g(y)$

$\frac{\partial F}{\partial y}\left(\frac{3}{2}x^2 + yx + g(y)\right)$

$= x + g'(y) = x$

$g'(y) = 0$

$g(y) = c_1$

$F(x,y) = \frac{3}{2}x^2 + xy + c_1$

② $\int B(x,y)\,dy = \int x\,dy$

$F(x,y) = xy + h(x)$

$\frac{\partial F}{\partial x} = y + h'(x) = 3x+y$

$h'(x) = 3x$

$h(x) = \frac{3}{2}x^2 + C_2$

$F(x,y) = xy + \frac{3}{2}x^2 + C_2$

$$\frac{3}{2}x^2 + xy = c$$

Inexact Equations: Integrating Factors

Similar to exact equations but $\frac{\partial A}{\partial y} \neq \frac{\partial B}{\partial x}$

can be made exact by multiplying by an integrating factor $\mu(x)$ s.t.

$$\frac{\partial(\mu A)}{\partial y} = \frac{\partial(\mu B)}{\partial x}$$

for integrating factor $\mu(x,y) = \mu$, no general method for solutions

there is a general method if integrating factor is function of either x or y alone

w/ $\mu = \mu(x)$

$$\frac{\partial(\mu A)}{\partial y} = \frac{\partial(\mu B)}{\partial x} \longrightarrow \mu\frac{\partial A}{\partial y} = \mu\frac{\partial B}{\partial x} + B\frac{d\mu}{dx}$$

$$\longrightarrow \mu\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right) = B\frac{d\mu}{dx}$$

$$\longrightarrow \frac{d\mu}{\mu} = \frac{1}{B}\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right)dx = f(x)\,dx$$

$$\ln(\mu) = \int f(x)\,dx$$

$$\mu = e^{\int f(x)dx}$$

w/ $\mu = \mu(y)$ $\rightarrow$ $\frac{\partial(\mu A)}{\partial y} = \frac{\partial(\mu B)}{\partial x} \rightarrow \mu \frac{\partial A}{\partial y} + A\frac{d\mu}{dy} = \mu\frac{\partial B}{\partial x}$

$$\rightarrow \mu\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) = A\frac{d\mu}{dy}$$

$$\rightarrow \frac{d\mu}{\mu} = \frac{1}{A}\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right)dy = g(y)\,dy$$

$$\ln(\mu) = \int g(y)\,dy$$

$$\mu = e^{\int g(y)dy}$$

if $\mu = \mu(x)$

$$\mu(x) = e^{\int f(x)dx} \quad w/ \quad f(x) = \frac{1}{B}\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right)$$

if $\mu = \mu(y)$

$$\mu(y) = e^{\int g(y)dy} \quad w/ \quad g(y) = \frac{1}{A}\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right)$$

Solve $\frac{dy}{dx} = \frac{-2}{y} - \frac{3y}{2x}$

$$2xy\,dy = -4x\,dx - 3y^2dx$$

$$(4x + 3y^2)\,dx + 2xy\,dy = 0$$

$$A(x,y) = 4x + 3y^2 \qquad \frac{\partial A}{\partial y} = 6y$$

$$B(x,y) = 2xy \qquad \frac{\partial B}{\partial x} = 2y \qquad \neq$$

$$\frac{1}{A}\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) \rightarrow \frac{1}{4x+3y^2}(2y - 6y) \rightarrow \frac{-3y}{4x+3y^2}$$ ew

$$\frac{1}{B}\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right) \rightarrow \frac{1}{2xy}(6y - 2y) \rightarrow \frac{4y}{2xy} \rightarrow \frac{2}{x}$$ nice :) f is of (x)

$$\mu(x) = e^{\int f(x)dx} = e^{\int \frac{2}{x}dx} = e^{2\ln(x)} = e^{\ln(x^2)} = x^2$$

$$(4x+3y^2)\,dx + 2x\,dy = 0$$

$$\xrightarrow{\cdot x^2} \quad (4x^3+3x^2y^2)\,dx + 2x^3\,dy = 0$$

$$M = 4x^3+3x^2y^2 = \frac{\partial F}{\partial x} \qquad \frac{\partial M}{\partial y} = 6x^2y$$
$$N = 2x^3y = \frac{\partial F}{\partial y} \qquad \frac{\partial N}{\partial x} = 6x^2y \qquad = \checkmark$$

go about exact steps

$$F(x,y) = \int (2x^3y)\,dy = x^3y^2 + h(x)$$

$$\frac{\partial F}{\partial x} = 3x^2y^2 + h'(x) = 4x^3 + 3x^2y^2$$

$$\longrightarrow h'(x) = 4x^3$$

$$h(x) = x^4 + C$$

$$F(x,y) = x^3y^2 + x^4 + C$$

$$\longrightarrow C = x^4 + x^3y^2$$

Solution Method

1. estimate $f(x)$ & $g(y)$ are f^n's of only x or y
2. find appropriate integrating factor
3. solve like an exact differential eq^n

Linear Equations

big picture: making something fit format of product rule & work backwards

Special case of inexact first order ODEs w/ form

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

need a specific integration factor

always f^n of x

simple form

$$\overbrace{\mu(x)}^{\mu(x)} \cdot \overbrace{\frac{dy}{dx}}^{\frac{d}{dx}[y]} + \overbrace{\mu(x)P(x)}^{\frac{d}{dx}[\mu(x)]}\,\overbrace{y}^{y} = \frac{d}{dx}\left[\mu(x)\,y\right] = \mu(x)\,Q(x)$$

can then be integrated s.t.

$$\mu(x)\,y = \int \mu(x)\,Q(x)\,dx$$

requirements for μ $\quad \frac{d}{dx}\left[\mu(x)\,y\right] = \mu\frac{dy}{dx} + \frac{d\mu}{dx}\,y = \mu\frac{dy}{dx} + \mu P(x)\,y$

$$\frac{d\mu}{dx} = \mu P(x) y$$

solution: $\mu(x) = e^{\int P(x)dx}$

Solve $\frac{dy}{dx} + 2xy = 4x$

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

$P(x) = 2x$ $\qquad \mu(x) = e^{\int 2x\,dx} = e^{x^2}$

$$\frac{dy}{dx} \cdot e^{x^2} + e^{x^2} \cdot 2x \cdot y = 4xe^{x^2}$$

$$\frac{d}{dx}\left[e^{x^2} \cdot y\right] = 4xe^{x^2}$$

$$e^{x^2} y = 4\int xe^{x^2}dx$$

$$e^{x^2} y = 2e^{x^2} + C$$

$$y = 2 + Ce^{-x^2}$$

$\int xe^{x^2}dx$
$\int \frac{1}{2} e^{u} du$
$\frac{1}{2}e^{u}$
$\frac{1}{2}e^{x^2}$

$u = x^2$
$du = 2x\,dx$
$\frac{du}{2} = x\,dx$

Solution Method
1. Rearrange
2. Multiply by $\mu(x)$
3. Integrate & solve

Homogeneous Equations

function is homogeneous of degree n if, for any λ, it obeys: $f(\lambda x, \lambda y) = \lambda^n f(x,y)$

$A = x^2y - xy^2 \rightarrow$ degrees: 3 $\qquad$ add exponents
$B = x^3 + y^3 \rightarrow$ degrees: 3

homogeneous equation are ODEs that can be in form

$$\frac{dy}{dx} = \frac{A(x,y)}{B(x,y)} = F\left(\frac{y}{x}\right)$$

w/ $A(x,y)$ & $B(x,y)$ are homogeneous f^{n}'s we say $y = vx$

$$\frac{dy}{dx} = \frac{d}{dx}(vx) = v \cdot \frac{dx}{dx} + x\frac{dv}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = F(v) \longrightarrow x\frac{dv}{dx} = F(v) - v$$

$$\longrightarrow \frac{dv}{F(v)-v} = \frac{dx}{x}$$

$$\longrightarrow \int \frac{dv}{F(v)-v} = \int \frac{dx}{x}$$

Solve $\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right)$

$y = vx \longrightarrow \cancel{v} + x\frac{dv}{dx} = \cancel{v} + \tan(v)$

$$\longrightarrow \frac{dv}{\tan(v)} = \frac{dx}{x}$$

$$\longrightarrow \int \cot(v)\,dv = \int \frac{dx}{x}$$

$\int \cot(v)\,dv$

$\int \frac{\cos(v)}{\sin(v)}dv$ $\quad u = \sin(v)$, $du = \cos(v)\,dv$

$\int \frac{du}{u} = \ln(u) + C_2 = \ln(\sin(v)) + C_2$

$\int \frac{dx}{x} = \ln(x) + C_1$

$\ln(\sin(v)) + C_2 = \ln(x) + C_1$

$\ln(\sin(v)) = \ln(x) + C_3$

$e^{\ln(\sin(v))} = e^{\ln(x) + C_3} = e^{\ln(x)} \cdot e^{C_3}$

$\sin(v) = x \cdot A$ $\qquad A = e^{C_3}$

$\sin\left(\frac{y}{x}\right) = Ax$

$\frac{y}{x} = \arcsin(Ax)$

$y = x\arcsin(A \cdot x)$

Solution Method

1. Check if equation is homogeneous
2. subst. $y = vx$, separate variables
3. integrate
4. $v = \frac{y}{x}$