defⁿ impt. — general — examples — titles — solution methods

previously saw Fourier series of a periodic f^n in a fixed interval as superposition of sin f^n's

often want representation over infinite interval w/no periodicity $\rightarrow$ Fourier Transform

13.1 Fourier Transforms

F.T. provides view of f^n's defined over infinite interval w/no particular periodicity

considered as generalization of Fourier series representation

b/c F.T. usually represent time-varying f^n's, usually use $f(t)$ rather than $f(x)$

only requirement: $\int_{-\infty}^{\infty} |f(t)|\, dt$ is finite

recall that a f^n of period T may be represented as complex F.S. (Fourier Series)

$$f(t) = \sum_{r=-\infty}^{\infty} c_r e^{\frac{2\pi i r t}{T}} = \sum_{r=-\infty}^{\infty} c_r e^{i\omega_r t} \qquad \omega_r = \frac{2\pi r}{T}$$

as $T \rightarrow \infty$, 'frequency quantum' $\left(\Delta\omega = \frac{2\pi}{T}\right)$ becomes vanishingly small

ω_r becomes a continuum

c_r coefficients $\rightarrow$ ω continuous variable

F.S. series $\rightarrow$ integral

recall how to get coefficients

$$c_r = \frac{1}{T}\int_{-T/2}^{T/2} dt\, f(t)\, e^{-\frac{i2\pi r t}{T}} = \frac{\Delta\omega}{2\pi}\int_{-T/2}^{T/2} dt\, f(t)\, e^{-i\omega_r t} = c_r$$

plug in c_r into $f(t)$

$$f(t) = \underbrace{\sum_{r=-\infty}^{\infty} \frac{\Delta\omega}{2\pi}\int_{-T/2}^{T/2} du\, f(u)\, e^{-i\omega_r u}}\; e^{i\omega_r t}$$

ω_r is still a discrete f^n of r

$$\omega(r) = \frac{2\pi r}{T}$$

$\frac{2\pi}{T} \cdot c_r \cdot e^{i\omega_r t}$ is area of r^{th} broken line

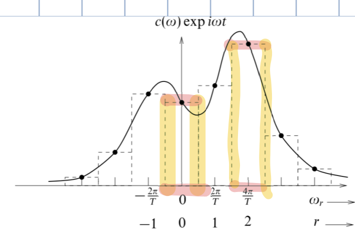


Figure 13.1 The relationship between the Fourier terms for a function of period T and the Fourier integral (the area below the solid line) of the function.

if $T \to \infty$, Δw becomes infinitesimal

by def^{n} of integral: $\sum_{r=-\infty}^{\infty} \frac{\Delta w}{2\pi} g(\omega_r) e^{i\omega_r t} \longrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} g(\omega)\, d\omega$

with: $g(\omega_r) = \int_{-T/2}^{T/2} f(u)\, e^{-i\omega_r u}\, du$

rewriting $f(t)$: $T \to \infty$ ① $-T/2 \to -\infty$ ② $T/2 \to \infty$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega t} \int_{-\infty}^{\infty} du\, f(u)\, e^{-i\omega u}$$

called Fourier's inversion th^{m}

can define the Fourier Transform of $f(t)$ $\tilde{f}(\omega)$ & its inverse $f(t)$

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt \qquad \text{eq } 1$$

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{f}(\omega)\, e^{i\omega t}\, d\omega \qquad \text{eq } 2$$

have $\frac{1}{\sqrt{2\pi}}$ to make sure products $= \frac{1}{2\pi}$, meant to be very symmetric

Find Fourier Transform of $f(t) = 0$ for $t < 0$ & $f(t) = Ae^{-\lambda t}$ for $t \geq 0$ + $\lambda > 0$

① $$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt\right)$$

$$= \frac{1}{\sqrt{2\pi}}\left(\underbrace{\int_{-\infty}^{0} 0 \cdot e^{-i\omega t}\, dt}_{0} + \int_{0}^{\infty} Ae^{-\lambda t} e^{-i\omega t}\, dt\right)$$

$$= \frac{A}{\sqrt{2\pi}}\int_0^{\infty} e^{-(\lambda + i\omega)t}\, dt$$

$$= \frac{A}{\sqrt{2\pi}} \left.\frac{e^{-(\lambda+i\omega)t}}{-(\lambda+i\omega)}\right]_0^{\infty} = -\frac{A}{2\pi(\lambda+i\omega)}\cdot\left(\frac{1}{e^{\infty}} - \frac{1}{e^{0}}\right)$$

$$= \frac{A}{2\pi(\lambda+i\omega)}$$ A is only the amplitude of transform

Uncertainty Principle

impt. f^{n} is Gaussian or normal distribution
its F.T. impt. b/c illustrates uncertainty principle

Find Fourier transform of normalized Gaussian distribution $f(t) = \frac{1}{\tau\sqrt{2\pi}} e^{-\frac{t^2}{2\tau^2}}$ $-\infty < t < \infty$

centred @ $t = 0$ & root mean squared deviation $\Delta t = \tau$

① $$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{\tau\sqrt{2\pi}} e^{-\frac{t^2}{2\tau^2}} e^{-i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{\tau\sqrt{2\pi}} e^{-\frac{1}{2\tau^2}t^2 - i\omega t}\, dt$$

$$-\frac{1}{2\tau^2}\left(t^2 + 2i\omega t\tau^2\right)$$

$a^2 + 2ab + b^2$

$$= -\frac{1}{2\tau^2}\left(t^2 + 2i\omega t\tau^2 + (\tau^2 i\omega)^2 - (\tau^2 i\omega)^2\right)$$

$$= -\frac{1}{2\tau^2}\left(t^2 + 2i\omega t\tau^2 - \tau^4\omega^2\right) - \frac{\tau^4\omega^2}{2\tau^2}$$

$$= -\frac{1}{2\tau^2}\left(t + i\tau^2\omega\right)^2 - \frac{\tau^2\omega^2}{2}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\tau\sqrt{2\pi}} \cdot e^{-\frac{1}{2\tau^2}(t+i\tau^2\omega)^2 - \frac{\tau^2\omega^2}{2}} dt$$

$$= \frac{e^{-\frac{\tau^2\omega^2}{2}}}{\sqrt{2\pi}} \underbrace{\frac{1}{\tau\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\tau^2}(t+i\tau^2\omega)^2} dt}$$

→ equals unity w/complex variable theory

$$\rightarrow \tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{\tau^2\omega^2}{2}}$$ } another Gaussian distribution!

root mean square deviation $\Delta\omega = \frac{1}{\tau}$

the F.T. of Gaussian is another Gaussian!

the root mean square deviation in t was τ, spreads in ω & t are inversely related

$$\Delta\omega \, \Delta t = 1$$ indpt. of τ

narrower in time → greater spread of frequency

for Gaussian wave packet: $\Delta k \, \Delta x = 1$

$p = \hbar k$ & $E = \hbar\omega$ → deBroglie & Einstein

in QM, $f(t)$ is wave function & distribution of wave intensity in time given by $|f|^2$ → a Gaussian! (Root mean square deviation of $\tau/\sqrt{2}$)

⇝ intensity distribution in frequency goes by $|\tilde{f}|^2$ (Root mean square deviation of $\frac{1}{\sqrt{2}\,\tau}$)

$\Delta E \, \Delta t = \hbar/2$ & $\Delta p \, \Delta x = \hbar/2$

Dirac δ-Function

can be visualized as a very sharp narrow pulse which produces integrated effect w/ a definite magnitude

2 big properties

① $\delta(t) = 0$ for $t \neq 0$

② $\int f(t)\,\delta(t-a)\,dt = f(a)$ provided range of integration has $t=a$, else equals zero

→ $\int_{-a}^{b} \delta(t)\,dt = 1$ for all $a, b > 0$

$\int \delta(t-a)\,dt = 1$ provided $t=a$ in integration range

→ $\delta(t) = \delta(-t)$

$\delta(at) = \frac{1}{|a|}\delta(t)$

$t\,\delta(t) = 0$

consider integral of form $\int f(t)\, \delta(h(t))\, dt$

$z = h(t) \longrightarrow \delta(h(t)) = \sum_i \frac{\delta(t-t_i)}{|h'(t_i)|}$

t_i are values where $h(t_i) = 0$ & $h'(t) = \frac{dh}{dt}$

derivative of $\delta(t)$:

$\delta(t-a)$ where $a = 0$

$$\int_{-\infty}^{\infty} f(t)\,\delta'(t)\,dt = \overbrace{\left[f(t)\,\delta(t)\right]_{-\infty}^{\infty}}^{\to\, 0} - \int_{-\infty}^{\infty} f'(t)\,\delta(t)\,dt$$

$$= -f'(0)$$

effects not strictly described by a δ fn can be analyzed as such if they take place in interval much shorter than response interval of system

Relation of δ-function to Fourier transforms

back to Fourier inversion Thm

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, e^{i\omega t} \int_{-\infty}^{\infty} du\, f(u)\, e^{-i\omega u} = \int_{-\infty}^{\infty} du\, f(u) \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(t-u)}\, d\omega \right\}$$

$\rightarrow$ can write δ-fn as ...

$$\int f(t)\, \delta(t-a)\, dt = f(a)$$

$$\delta(t-u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(t-u)}\, d\omega$$

very narrow time peak @ $t = u$ results from superposition of complete spectrum of harmonic waves, all frequencies w/same amplitude & all waves in phase @ $t = u$

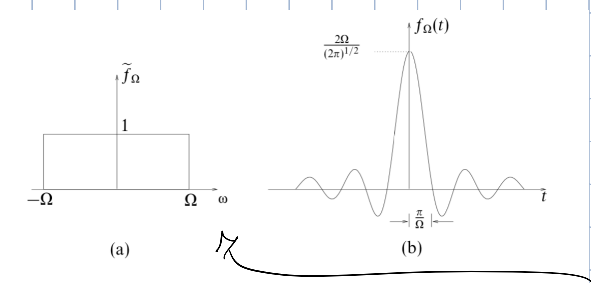


Figure 13.4 (a) A Fourier transform showing a rectangular distribution of frequencies between $\pm\Omega$; (b) the function of which it is the transform, which is proportional to $t^{-1}\sin\Omega t$.

https://commons.wikimedia.org/wiki/File:Fourier_transform_time_and_frequency_domains_(small).gif#/media/File:Fourier_transform_time_and_frequency_domains_(small).gif

consider distribution of frequencies, taking inverse Fourier Transform

$$f_\Omega(t) = \frac{1}{\sqrt{2\pi}} \int_{-\Omega}^{\Omega} 1 \times e^{i\omega t}\, d\omega$$

$$= \frac{2\Omega}{\sqrt{2\pi}} \frac{\sin(\Omega t)}{\Omega t}$$

large Ω, large & narrow @ $t = 0$

can represent δ fn as: $\delta(t) = \lim_{\Omega \to \infty} \left(\frac{\sin(\Omega t)}{\pi t} \right)$

$\int f(t)\, \delta(t-a)\, dt = f(a)$

F.T. of δ is real b/c $\quad \delta^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\, d\omega = \delta(-t) = \delta(t)$

$f(t) = e^{-a\ldots} \quad a = 0 \quad e^0 = 1$

F.T. of δ-f^n: $\quad \tilde{\delta}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(t)\, e^{-i\omega t}\, dt = \frac{1}{\sqrt{2\pi}} \cdot 1$

Properties of Fourier Transforms

F.T. of $f(t)$ is $\tilde{f}(\omega)$ or $\mathcal{F}[f(t)]$

Differentiation

$$\mathcal{F}[f'(t)] = i\omega\, \tilde{f}(\omega)$$

$$\mathcal{F}[f''(t)] = i\omega\, \mathcal{F}[f'(t)] = -\omega^2\, \tilde{f}(\omega)$$

Integration

represents F.T. of constant of integration

$$\mathcal{F}\left[\int^t f(s)\, ds\right] = \frac{1}{i\omega}\tilde{f}(\omega) + 2\pi c\, \delta(\omega)$$

Scaling

$$\mathcal{F}[f(at)] = \frac{1}{a}\tilde{f}\left(\frac{\omega}{a}\right)$$

Translation

$$\mathcal{F}[f(t+a)] = e^{ia\omega}\, \tilde{f}(\omega)$$

Exponential multiplication

$$\mathcal{F}[e^{\alpha t} f(t)] = \tilde{f}(\omega + i\alpha) \qquad \alpha \to \mathbb{R} \text{ or } \mathbb{I} \text{ or } \mathbb{C}$$

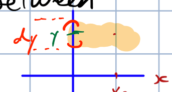
Convulation and Deconvulation

any attempt to measure physical quantity is limited by finite resolution of measuring apparatus

① Physical quantity (x) to measure ($f(x)$)

② Measuring apparatus doesn't give out true value ($f(x)$), resolution f^n is used ($g(y)$)

probability that output y=0 recorded between y & y+dy given by $g(y)\,dy$



good results, close to δ-f^n

typical apparatus

can be some bias (some systematic error)

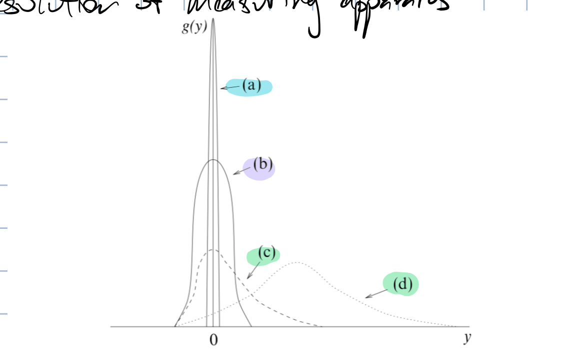


Figure 13.5 Resolution functions: (a) ideal δ-function; (b) typical unbiased resolution; (c) and (d) biases tending to shift observations to higher values than the true one.

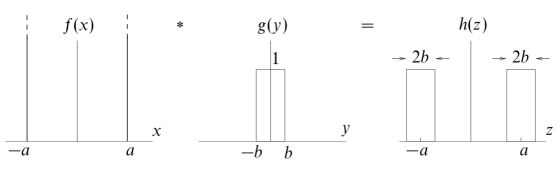


Figure 13.6 The convolution of two functions $f(x)$ and $g(y)$.

w/ true distribution $f(x)$ & resolution fn $g(y)$
want to calculate observed distribution $h(z)$

x, y, z all refer to same physical variable, but are analyzed differently

Probability true reading lying between x & $x+dx$ (having $f(x)dx$ probability of being selected by experiment) will be moved by instrumental resolution by $z - x$ into dz is $g(z-x)\,dz$

↳ probability for x being changed by measurement

Combined probability of dx giving observation appearing in dz is $f(x)\,dx \cdot g(z-x) \cdot dz$

Add all contributions of all x landing in range z & $z+dz$

→ $h(z) = \int_{-\infty}^{\infty} f(x)\, g(z-x)\, dx$ "convolution" of f & g

$f * g = g * f$

Observed distribution is convolution of true distribution & experimental resolution fn

convolution of any fn $g(y)$ w/ a number of δ-fn leaves a copy of $g(y)$ @ position of each δ-fn