def^n impt.

general
examples

titles
solution methods

12.1 Introduction

Some DE's can't be solved w/elementary f^n's

DE's will be linear w/coefficients as f^n's of x : $y'' + f(x)\,y' + g(x)\,y = 0$

Example 1

$$y' = 2xy$$

assume sol^n is of power series form

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n = \sum_{n=0}^{\infty} a_n x^n$$

$$\rightarrow \quad y' = a_1 + 2a_2 x + 3a_3 x^2 + \dots + n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$a_1 + 2a_2 x + 3a_3 x^2 + \dots + n a_n x^{n-1} = 2x \cdot (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n)$$

$$a_1 + 2a_2 x + 3a_3 x^2 + \dots + n a_n x^{n-1} = 2a_0 x + 2a_1 x^2 + 2a_2 x^3 + 2a_3 x^4 + \dots + 2a_n x^{n+1}$$

$$\rightarrow \quad a_1 = 0$$
$$a_2 = a_0$$
$$a_3 = \tfrac{2}{3} a_1 = 0$$
$$a_4 = \tfrac{1}{2} a_0$$

in general: $n a_n = 2 a_{n-2}$

$$a_n = \begin{cases} 0 & \text{odd } n \\ \frac{2}{n} a_{n-2} & \text{even } n \end{cases}$$

allow $n = 2m$

$$a_{2m} = \frac{2}{2m} a_{2m-2} = \frac{1}{m} a_{2m-2} = \frac{1}{m}\frac{1}{m-1} a_{2m-4} = \dots = \frac{1}{m!} a_0$$

$$y = a_0 + a_0 x^2 + \tfrac{1}{2} a_0 x^4 + \dots + \tfrac{1}{m!} a_0 x^{2m} = \sum_{m=0}^{\infty} a_0 \frac{x^{2m}}{m!}$$

if there is a solution which can be represented by a convergent power series, this finds it

12.2 Legendre's Equation

Legendre DE: $(1-x^2)\,y'' - (2x)\,y' + \ell(\ell+1)\,y = 0$

constant

pops up in QM, EM, heat...

most useful sol'ns are polynomials $\longrightarrow$ Legendre Polynomials

One way to find them is to assume a series sol'n & show series terminates after finite # of terms

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \dots + n a_n x^{n-1}$$

$$y'' = 2a_2 + 6a_3 x + 12 a_4 x^2 + \dots + n(n-1) a_n x^{n-2}$$

from Legendre

	constant	x	x^2	x^3	x^n
y''	$2a_2$	$6a_3$	$12a_4$	$20a_5$	$(n+2)(n+1)a_{n+2}$
$-x^2y''$			$-2a_2$	$-6a_3$	$-n(n-1)a_n$
$-2xy'$		$-2a_1$	$-4a_2$	$-6a_3$	$-2na_n$
$\ell(\ell+1)y$	$\ell(\ell+1)a_0$	$\ell(\ell+1)a_1$	$\ell(\ell+1)a_2$	$\ell(\ell+1)a_3$	$\ell(\ell+1)a_n$

Set total coefficient of each power of x equal to zero

$2a_2 + \ell(\ell+1)a_0 = 0$ or $a_2 = \frac{-\ell(\ell+1)}{2} a_0$

$6a_3 + (\ell^2+\ell-2)a_1 = 0$ or $a_3 = -\frac{(\ell-1)(\ell+2)}{6} a_1$

$12a_4 + (\ell^2+\ell-6)a_2 = 0$ or $a_4 = -\frac{(\ell-2)(\ell+3)}{12} a_2 = \frac{\ell(\ell+1)(\ell-2)(\ell+3)}{4!} a_0$

$$\underbrace{(n+2)(n+1)a_{n+2}}_{} + \underbrace{(\ell^2+\ell-n^2-n)}_{} a_n = 0$$

$$\ell^2 - n^2 + \ell - n = (\ell-n)(\ell+n) + (\ell-n) = (\ell-n)(\ell+n+1)$$

$$a_{n+2} = -\frac{(\ell-n)(\ell+n+1)}{(n+2)(n+1)} a_n$$

$$\longrightarrow y = a_0\left[1 - \frac{\ell(\ell+1)}{2}x^2 + \frac{\ell(\ell+1)(\ell-2)(\ell+3)}{4!}x^4 - \dots\right] + a_1\left[x - \frac{(\ell-1)(\ell+2)}{3!}x^3 + \frac{(\ell-1)(\ell+2)(\ell-3)(\ell+4)}{5!}x^5 + \dots\right]$$

12.6 Complete Sets of Orthogonal Functions

2 vectors $\vec{A}$ & $\vec{B}$ are $\perp$ if : $\sum_i A_i B_i = 0$

can say 2 fn's $A(x)$ & $B(x)$ are orthogonal on (a,b) if : $\int_a^b A(x)\, B(x)\, dx = 0$

if $A(x)$ & $B(x)$ are complex :

$A(x)$ & $B(x)$ are orthogonal on (a,b) if

$$\int_a^b A^*(x) B(x) dx = 0$$

$A^*(x)$ is complex conjugate of $A(x)$

if we have set of fn's $A_n(x)$, $n = 1, 2, 3,$ & $\int_a^b A_n^*(x) A_m(x)\, dx \begin{cases} 0 \text{ if } m \neq n \\ \text{const.} \neq 0 \text{ if } m = n \end{cases}$

$\longrightarrow$ $A_n(x)$ is a set of orthogonal fn's

Use of sets of fn's : Fourier series

$$\int_{-\pi}^{\pi} \sin(nx) \sin(mx)\, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \neq 0 \end{cases}$$

$\longrightarrow$ $\sin(nx)$ is a set of orthogonal fn's on $(-\pi, \pi)$

$\longrightarrow$ same w/ $\cos(nx)$

$$\int_{-\pi}^{\pi} \sin(nx) \cos(mx)\, dx = 0$$ for any m, n

$\longrightarrow$ whole set of $\cos(nx)$ & $\sin(nx)$ is set of orthogonal fn's

$$\int_{-\pi}^{\pi} (e^{inx})^* (e^{imx})\, dx = \int_{-\pi}^{\pi} e^{-inx} e^{imx}\, dx = \begin{cases} 0 & \text{if } m \neq n \\ 2\pi & \text{if } m = n \end{cases}$$

<u>complete</u> set if no other fn's orthogonal to all of them