Cheat Sheet for ASTR 204: Fundamentals of Astrophysical Fluid Dynamics
UC Santa Cruz, Fall Quarter 2023

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Contents

Section 0: Introduction

Contributors: *Diego Garza*

This was my first course at UC Santa Cruz. The lectures were all in-person, with Ruth primarily using her iPad projected onto a whiteboard. We had ∼weekly problem sets which usually had a coding component. The Github included helpful instructions on getting started with Github. We primarily used Canvas for assignment submissions, for her to share notes, and to work through modules. Our primary means of communication were either email or Discord. We had weekly office hours that were very helpful as well.

I really appreciated her lecture style, giving us problems to work on, and allowing for awkward silences to think about a problem. I particularly enjoyed the oral final that well-represented the course topics. From this course, I became much more comfortable with order of magnitude estimate calculations.

To create this cheat sheet, I often consulted Courtney Carreira's helpful cheat sheet, and asked plenty of clarification questions that my classmates helped with. Many thanks to my first-year cohort: Pedro-Jesus Quiñonez, Malik Bossett, Mika Lambert, Anna Gagnebin, Lordrick Kahinga, and Courtney Carreira.

A note on convention. I often use ∂_i^j to represent the partial differential to some order $\frac{\partial^j}{\partial i^j}$, and likewise $d_i^j = \frac{d^j}{di^j}$. For example, $\partial_t^2 = \frac{\partial^2}{\partial t^2}$ and $d_x^3 = \frac{d^3}{dx^3}$. This is to help write in-line equations at times.

Helpful links as I was taking the class:

- Physical Constants and Astronomical Data
- Del in cylindrical and spherical coordinates
- Vector calculus identities

Section 1: Fluid Mechanics Theory

Contributors: *Diego Garza*

Note: *LaTeX format adapted from template for lecture notes from CS 267, Applications of Parallel Computing, UC Berkeley EECS department.*

1.1 What is a Fluid?

A fluid is an approximation of particles, which are part of a larger system, that can be well-described by fields. There are a couple requirements to this for a blob of particles to be considered a fluid element part of a larger system.

1. The number of particles in the system is much greater than 1 $N_{\text{tot}} \gg 1$
2. The number of particles *in a fluid element* is much greater than the total number of particles in the system $N \ll N_{\text{tot}}$
3. We have a well defined set of thermodynamic variables including
   - Pressure - P
   - Volumetric Density - $\rho \equiv M/V$
   - Velocity - $\vec{v}$
   - Volume - V
   - Temperature - T
4. Characteristic size of the system is much greater than the mean free path, or the distance a particle expects to travel before colliding with another particle $L \gg \lambda_{\text{mfp}}$
5. Characteristic time scale of the system that describes how long an overall system property to change is much greater than the collision time, or time a particle expects to travel before colliding with another particle $\tau \gg t_{\text{coll}}$

1.2 Collisional Rates

With a fluid element, we can approximate the rate at which collisions occur with three properties

1. The volumetric number density of particles that may collide - n
2. The cross-section of the collisions - σ
3. The speed at which particle constituents are moving at - v. This is *usually* the thermal velocity v_{th}, but can also be the bulk velocity v_{bulk}. It is situation specific.

The thermal velocity v_{th} is a statistical description of fluid elements as a Maxwellian Distribution. There are different conventions on how to define v_{th} from the distribution, but is generally well-approximated as the root mean square of the velocity in one dimension where we can say $v_{\text{th}} \sim \sqrt{k_B T/m}$ with k_B as the Boltzmann constant and m representing the mass of a single particle. In this case, we can approximate the thermal velocity as the sound speed of a fluid $v_{\text{th}} \sim c_{\text{s}}$.

The collisional rate is calculated by multiplying all three properties $n\sigma v$. For a fluid element, this is a useful characteristic value.

1.3 Conservation Equations

All physics is just conservation equations. There are two ways to define a set perspective to view a value's conservation:

- Lagrangian: Fixed in volume. We consider one volume element and move with it.
- Eulerian: Fixed in space. We consider one volume in space and view what goes in and out.

In general, to convert from one to another, we can use the fact that the Lagrangian perspective makes use of the Material Derivative (also called the Lagrangian or Total Derivative)

$$\frac{D}{Dt} = \frac{d}{dt} = \frac{\partial}{\partial t} + \vec{u} \cdot \mathbf{\nabla} \tag{1.1}$$

where $\vec{u}$ is the macroscopic, drift, or flow velocity. In general, I will be using the Eulerian perspective unless otherwise specified.

1.3.1 Mass Conservation

The primary idea is that we can find the total mass change in a fluid element by considering the mass flow into and out of the element.

$$\frac{\partial \rho}{\partial t} + \vec{\mathbf{\nabla}} \cdot (\rho \vec{v}) = 0 \tag{1.2}$$

In the Lagrangian perspective

$$\frac{d\rho}{dt} + \rho \vec{\mathbf{\nabla}} \cdot \vec{v} = 0$$

There are no sources/sinks of mass

1.3.2 Momentum Conservation

For momentum conservation, things get slightly complicated because we have to consider how the momentum's velocity changes with the flow velocity of the fluid. These aren't necessarily in the same direction. We can have momentum in one direction moving in another direction.

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j) = -\frac{\partial P}{\partial x_i} + \frac{\partial \phi}{\partial x_j} + \rho f_i \tag{1.3}$$

where the first term is the time derivative of the momentum density and the second term describes momentum flux or how the momentum flux tensor changes. The right hand term handles the element's own pressure and any source/sink with ϕ and external terms in that i-direction with f_i serving as the force per unit mass. If important for the problem, viscosity is encapsulated as an internal source/sink of momentum in ϕ

The Lagrangian perspective is much more compact but provides the same picture.

$$\rho \frac{d\vec{v}}{dt} = -\vec{\boldsymbol{\nabla}} P + \vec{\boldsymbol{\nabla}} \phi + \rho \vec{f}$$

This is very analogous to the traditional Newton's Second Law of Motion $\vec{F} = m\vec{a}$

1.3.3 Energy Conservation

From the First Law of Thermodynamics, we can relate entropy, internal energy, and work done by internal pressure as $dU = TdS - PdV$. Furthermore, the conservation of energy can be described as

$$\frac{\partial E}{\partial t} + \vec{v} \cdot \vec{F}_E = \rho \boldsymbol{\nabla} \phi \cdot \vec{v} + \rho \vec{f} \cdot \vec{v} \tag{1.4}$$

where again $\vec{f}$ is the force per unit mass and $\vec{F}_E$ is the energy flux vector, which is equivalent to

$$\vec{F}_E = E\vec{v} + P\vec{v} = \frac{1}{2}\rho v^2 \vec{v} + (\varepsilon + P/\rho)\vec{v}$$

In a Lagrangian perspective

$$\rho T \frac{ds}{dt} = \rho \boldsymbol{\nabla} \phi + \rho f$$

where $\boldsymbol{\nabla}\phi$ encapsulates all internal sources and sinks of energy. Within the context of astrophysical fluid dynamics, the sources/sinks and external forces are generally well described by $\rho \boldsymbol{\nabla} \phi + \rho f = \Psi - \vec{\boldsymbol{\nabla}} \cdot \vec{F}_C + \Gamma - \Lambda$, which incorporates viscosity, conduction, radiative heating, and radiative cooling.

For an isotropic system, $d_t(s) = 0$, which results in a relationship between pressure and density as $P \propto \rho^\gamma$, with γ as the adiabatic index. A more detailed discussion on energy conservation can be found in Section 2.1.

1.4 Diffusion

Diffusion is the movement of particles from a high to low concentration through "random walks". Under the fluid element assumptions, particles moving will experience a collision, causing it to "walk" in a random direction.

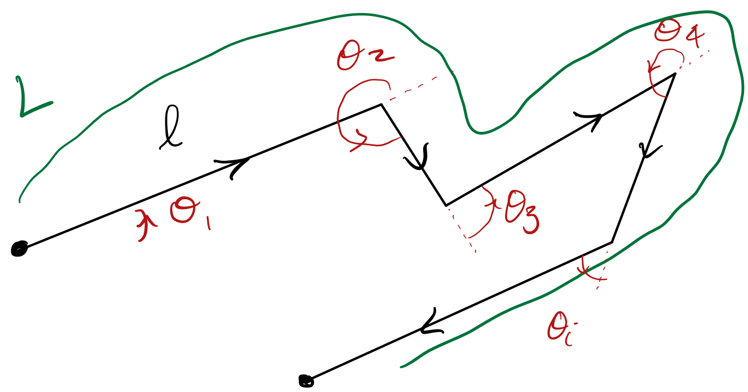

Figure 1.1: Particle experiencing collisions and changes in orientation of it's velocity.

A particle experiencing N number of steps/walks, each with about the same length between collisions l, will travel a total distance of L, as demonstrated in Figure 1.1.

Asssuming the change in orientations, θ_i in Figure 1.1, are all independent from one another (hence "random"), the average length between collisions and the total distance traveled are related as

$$L = l\sqrt{N} \tag{1.5}$$

Just like we related the average length between collisions and the total length traveled, we can similarly define a characteristic time between collisions (Δt) and the total time elapsed (τ). We can then describe the number of steps taken with the time element

$$N = \frac{\Delta t}{\tau}$$

We can rewrite 1.5 with this information, to describe the macroscopic length scale and the macroscopic time scale

$$L^2 = \frac{l^2}{\Delta t}\tau = D\tau$$

where we define the diffusion coefficient

$$D = \frac{l^2}{\Delta t} = lv$$

with the microscopic velocity between collisions $v = l/\Delta t$. However, the diffusion coefficient can similarly be described using the macroscopic quantities. In effect, the diffusion coefficient is, in general

$$D = \frac{L^2}{T} = LV = V^2 T \tag{1.6}$$

where either ALL of the quantities (V, L, T) are of the *macroscopic* description of the entire system OR the *microscopic* description of the collisions.

For the macroscopic quantities

- L - total length traveled by a particle, characteristic length scale of system
- $T = \tau$ - total time elapsed for particle to travel, characteristic time scale of system
- V - characteristic velocity of entire system, $V = L/\tau$

An important result of random walk diffusion is how all three are related to one another, particularly $V = \sqrt{D/\tau} \propto t^{-1/2}$

The microscopic quantities are statistical averages of collisional properties

- $L = l$ - average length traveled by a particle before collision
- $T = \Delta t$ - average time elapsed for particle to travel before a collision
- $V = v$ - average velocity of particles traveling before collision

For our astrophysical fluid dynamics applications, diffusion collisions can be described by using the average length traveled before a collision as the mean free path $l = \lambda_{\rm mfp}$, the average time elapsed before a collision as the collision time $\Delta t = t_{\rm coll}$, and the average velocity $v = v_{\rm th}$.

For a quantity $f(x,t)$ that is diffusing along the x direction, the diffusion equation is defined as

$$\frac{df}{dt} = D\frac{d^2 f}{dx^2} \tag{1.7}$$

For a field that satisfies this form, the diffusion coefficient D tells us about the relationship between microscopic collisional properties ($\lambda_{\rm mfp}, t_{\rm coll}, v$) and the relationship between macroscopic system properties (L, τ, V).

1.5 Streamlines

A streamline is defined as the path of a fluid such that the tangent to the streamline is the velocity of the fluid. This is very helpful in cases where we have steady states

$$\frac{\partial}{\partial t} = 0$$

With this assumption, the conservation equations are greatly simplified as the Lagrangian derivative reduces down to

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{u} \cdot \boldsymbol{\nabla} = \vec{u} \cdot \boldsymbol{\nabla}$$

With a steady-state system, we are looking *along* the direction of the velocity, there are no perpendicular components.

Section 2: Thermodynamics Theory

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2.1 Fundamental Relation of Thermodynamics

The conservation of energy is already pretty well-described, but I would like to provide a bit of motivation from the First Law of Thermodynamics

$$dU = TdS - PdV$$

The statement describes that the change in internal energy is equivalent to the change in heat energy subtracted by the internal pressure-volume work. The internal energy of an ideal gas with a total number of N particles can be described as

$$U = N\frac{l}{2}k_BT$$

in which the temperature is well defined, and l is the number of degrees of freedom that a particle has.

When discussing the dynamics of a system, we (lol, I am a dynamicist now) usually prefer to speak in *specific* quantities, or quantities normalized by the mass of a particle. Only extensive quantities (properties that scale with the size of the system) are spoken about per mass, which include energy and entropy. Intensive quantities (properties which *do not* scale with size) are not effected, which include temperature and pressure. The First Law of Thermodynamics then becomes

$$dU/M = TdS/M - PdV/M \rightarrow d\varepsilon = Tds - PdV/M = Tds + \frac{P}{\rho^2}d\rho \tag{2.1}$$

where we take advantage of the fact that $\rho = M/V$.

2.2 Adiabatic Index

The adiabatic index is defined as

$$\gamma = \frac{c_\text{P}}{c_\text{V}} \tag{2.2}$$

where the numerator is the heat capacity at constant pressure and the denominator is the heat capacity at constant volume. The heat capacity is calculated using

$$c_i = \left(\frac{đQ}{dT}\right)\bigg|_i \tag{2.3}$$

For an ideal gas, the heat capacities are found as $c_\text{P} = (l/2)k_B N + Nk_B$ and $c_\text{V} = (l/2)Nk_B$, which reduces the adiabatic index down to

$$\gamma = \frac{l+2}{l} \tag{2.4}$$

Using the mean average mass of a particle μ, we can describe the specific internal energy of an ideal gas using either the degrees of freedom or the adiabatic index

$$\varepsilon = \frac{N}{M}\frac{l}{2}k_B T = \frac{1}{\mu}\frac{1}{\gamma - 1}k_B T \tag{2.5}$$

2.3 Conservation of Energy

We can set the total energy of a system as the sum of the kinetic energy and the internal energy

$$E = \frac{1}{2}\rho v^2 + \rho\varepsilon \tag{2.6}$$

In a similar vein as the mass and momentum conservation discussions, we show the energy conservation in the Eulerian perspective

$$\frac{\partial E}{\partial t} + \vec{\boldsymbol{\nabla}} \cdot ((E+P)\vec{v}) = \rho \vec{f} \cdot \vec{v} \tag{2.7}$$

where we bring the PdV work onto the left-hand side and the right-hand side describes any external work done onto the system. Writing the equation completely out and assuming no other internal sources/sinks of energy other than the pressure (setting $\boldsymbol{\nabla}\phi = 0$), we find

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho\varepsilon\right) + \vec{\boldsymbol{\nabla}} \cdot \left(\vec{v}\left(\frac{1}{2}\rho v^2 + \rho\varepsilon\right) + P\vec{v}\right) = \rho \vec{f} \cdot \vec{v}$$

The energy flux vector is defined in terms of the combination of the total energy and pressure

$$\vec{F}_E = (E+P)\vec{v}$$

Another helpful perspective is to notice that the internal energy can change with the change in pressure and volume, which provides motivation for enthalpy

$$H = U + PV$$

or in specific quantities

$$h = \varepsilon + \frac{P}{\rho} \tag{2.8}$$

This can help provide a new perspective on the energy flux vector

$$\vec{F}_E = \vec{v}(E + P) = \vec{v}\frac{1}{\rho}\left(\frac{1}{2}v^2 + \varepsilon + \frac{P}{\rho}\right) = \vec{v}\frac{1}{\rho}\left(\frac{1}{2}v^2 + h\right) \tag{2.9}$$

so it includes a contribution from the flux of kinetic energy and flux of enthalpy. With a fixed pressure, there is a trade off in the fluid element's PdV work and change in internal energy unless the kinetic energy (from bulk flow) or external forces transfer additional energy into the internal energy.

An isentropic and adiabatic system is describd as $d_t(s) = 0$.

Section 3: Wave Mechanics Theory

Contributors: *Diego Garza*

3.1 General Solution to Wave Equation

The general wave equation for some scalar quantity $\phi(x,t)$ in one dimension is

$$\frac{\partial^2\phi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\phi}{\partial t^2} \tag{3.1}$$

which can be generalized to three dimensions with the Laplacian operator

$$\boldsymbol{\nabla}^2\phi = \frac{1}{v^2}\frac{\partial^2\phi}{\partial t^2} \tag{3.2}$$

where v is the speed at which the wave is traveling at. We can further simplify this equation by introducing the d'Alambert operator (also known as the wave operator)

$$\Box = \frac{1}{v^2}\frac{\partial}{\partial t} - \boldsymbol{\nabla}^2 \tag{3.3}$$

such that our wave equation is now simply $\Box\phi = 0$. The plane-wave solution (which we are most interested in) in one dimension is given by

$$\phi(x,t) = A\sin(kx - \omega t) = A\sin(k(x - vt)) \tag{3.4}$$

With this plane-wave solution, we can define and further describe some helpful quantities

- A - amplitude
- ω - angular frequency (radians per time)
- k - angular mode
- $T = 2\pi/\omega$ - period
- $\nu = 1/T$ - frequency (cycles per time)
- $\lambda = 2\pi/k$ - wavelength
- $v = \omega/k = \lambda/T$ - phase velocity
- $v_g = \partial_k(\omega)$ - group velocity

The dispersion relationship is the relationship between ω and k. The plane-wave solution's description of the phase velocity is a *linear* dispersive relationship. If the dispersive relationship includes an imaginary component, it is described as "dispersive" (Not 100% clear, would like a better explanation).

3.2 Perturbation Theory on Waves

Helpful in lots of order of magnitude calculation is learning when and how perturbations to a known solution propagates. With a known solution ϕ_0, we would like to consider a small perturbation ϕ_1 of order $\lambda \ll 1$

$$\phi(x,t) = \phi_0(x,t) \rightarrow \phi(x,t) = \phi_0(x,t) + \lambda\phi_1(x,t)$$

ϕ_0 *used to* be the correct full description of the quantity ϕ, but some disturbance has changed it. We study these perturbations to a plane-wave solution with Fourier Series Decomposition, where we require

$$e^{i\phi} = \cos(\phi) + i\sin(\phi) \tag{3.5}$$

The perturbation is described as

$$\phi_1 = \delta\phi \exp\Big[i(\vec{k}\cdot\vec{x} - \omega t)\Big]$$

If some wave vector $\delta\vec{\Lambda}$ and the wave mode vector $\vec{k}$ are along the same direction, then we know the perturbation of that wave vector is along the direction of travel.

Section 4: Keplerian Dynamics and Gravity

Contributors: *Diego Garza*

4.1 Kepler's 3 Laws

The Two-Body problem encapsulates Kepler's Three Laws of planetary motion. The Two-Body problem aims to solve the equation

$$\frac{d^2\vec{r}}{dt^2} = -\frac{Gm_1 m_2}{r^2}\hat{r} \tag{4.1}$$

In the barycentric (intertial) frame, we have four constants: energy per reduced mass (ε) and angular momentum per reduced mass ($\vec{h}$). Solving in polar coordinates, we find

$$\frac{dr}{dt} = \pm\left(2(\varepsilon + \frac{G(m_1+m_2)}{r}) - \frac{h^2}{r^2}\right) \text{ and } \frac{d\theta}{dt} = \frac{h}{r^2}$$

To have some time evolution, we can take the inverse of the first equation

$$t = \int \left(\frac{dr}{dt}\right)^{-1} dr$$

Getting rid of the time dependence, we find the solution to be a description of an ellipse on a conic section

$$r = \frac{p}{1 + e\cos(\theta - \varpi)} \tag{4.2}$$

where $p = h^2/(G(m_1 + m_2))$, e is the eccentricity, and ϖ is an integration constant which all come from solving the second-order differential equation.

For the Two-Body problem with the two bodies being gravitationally bound to one another, the two bodies will orbit in an elliptical orbit, with the most massive sitting at a focus. Kepler's First Law (Sun is at a focus with planets moving in ellipses) is an example of this. Kepler's Second Law says that, in an equal amount of time, the smaller body will "sweep out" the same amount of area of the ellipse. In Figure 4.2, the left hand part of the orbit will move slower and the right hand will move faster, but both will sweep out the same amount of area given the same amount of time.

Regarding ellipses and the Two-Body problem, there are many other important topics that can be covered including: conic sections with eccentricities that aren't an ellipse (where $0 < e < 1$ doesn't hold), orbital periods (including how we define a dynamical time), orbits in time (including mean and eccentric anomaly), orbits in space (where our frame of reference is in the xy plane and the ellipse is at an incline of i), the Guiding Center approximation (useful for circular orbits in a rotating frame), and axisymmetric potentials. However, for our purposes, we just want to define a couple key properties that can be used

- eccentricity e defines how circular ellipse is. $e = 0$ corresponds to a circle and $e = 1$ describes a straight, infinite line

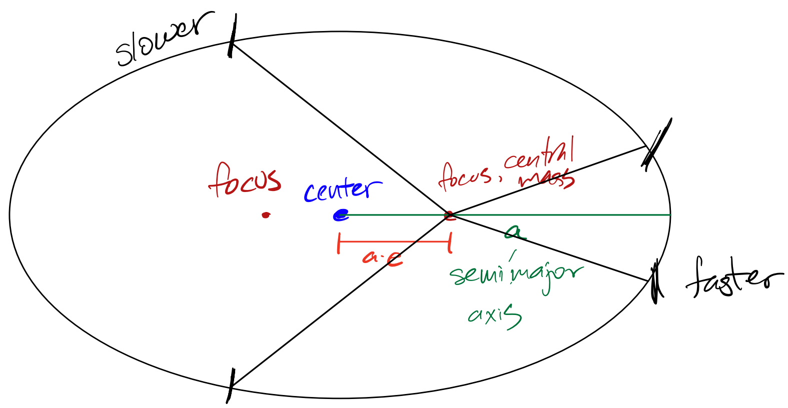

Figure 4.2: Example diagram of a Keplerian orbit as an ellipse

- pericenter $r_{\text{peri}} = a(1-e)$ is the *closest* approach distance from the central mass focus to the ellipse
- apocenter $r_{\text{apo}} = a(1+e)$ is the *furthest* approach distance from the central mass focus to the ellipse
- tangential velocities
  - velocity at pericenter $v_{\text{peri}} = v_k\sqrt{(1+e)/(1-e)}$
  - velocity at apocenter $v_{\text{apo}} = v_k\sqrt{(1-e)/(1+e)}$
  - comparison of the three velocities $v_{\text{apo}} < v_k < v_{\text{peri}}$

where we apply Kepler's Third Law to define v_k. Kepler's Third Law relates the orbital period and the size of the orbit by the semi-major axis a. In the Two-Body problem with a large central mass, this is described as

$$GM = a^3\Omega_k^2 = v_k^2 a \tag{4.3}$$

The k subscripts on the orbital frequency Ω and the velocity v are to denote that these describe a Keplerian orbit. We can also note

$$v_k = \sqrt{\frac{GM}{a}}$$

to compare the different tangential velocities from earlier. A version of Kepler's Third Law is found with a circular orbit, setting the centrifugal acceleration equal to gravitational acceleration

$$\frac{GM}{r^2} = \frac{v^2}{r} \rightarrow GM = v^2 r = \Omega^2 r^3$$

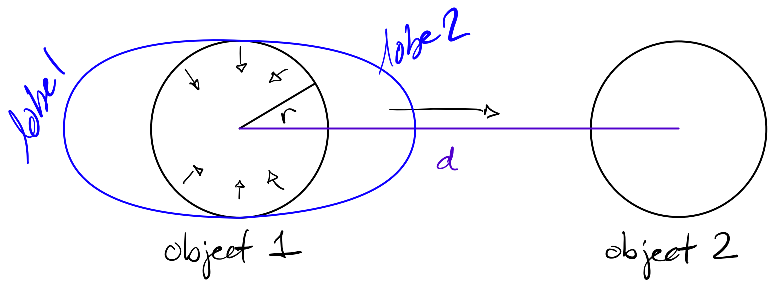

Figure 4.3: Example of a two-body tidal interaction

4.2 Tidal Gravity

Tidal gravity describes the spatial distribution of gravity for two bodies. Body 1 has its own self-gravity

$$F_g = \frac{GM_1}{r^2}$$

where we normalize by some test particle's mass that sits on its surface. M_1 describes the body's mass and r is the distance to that test particle. In the presence of a second body, Body 1 also experiences tidal force

$$F_{\text{tid}} = \frac{GM_2}{d^3} r \tag{4.4}$$

where the second body's mass is M_2 and the distance separated between the two bodies is d. This is essentially a second-order correction to account for other objects. In Figure 4.3, we see that the right-hand Lobe 2 is being slightly more attracted to Body 2 than the rest of Body 1. Correspondingly, Lobe 1 is the effect of all other mass in Body 1 being pulled because it is closer to Body 2, so it is slightly left behind and trying to catch up.

The second body also experiences tidal acceleration from the first body, but the relative strength is dependent on where the test particle is and the relative masses M_1 and M_2.

Section 5: Applications

Contributors: *Diego Garza*

Applications of these topics include

1. Hydrostatic Equilibrium
2. Streamlines and the shape of airplane wings
3. Stellar (Parker) winds
4. Sound Waves in Atmosphere
5. Viscosity as a Diffusive force
6. Thermal Conduction as a Diffusive force
7. Accretion Disks
   - Criteria for a rotationally supported disk
   - Criteria to ignore rotation
   - Viscous transport of angular momentum
8. Gravitational collapse
9. Tidal forces ripping apart smaller body
10. How long takes to cook a turkey

These applications and extensions of them were worked on during class and in problem sets.

References