Charlies notes & Lecture 16

$$\partial = \Box = \nabla^2 - \frac{1}{c^2}\partial_t^{\,2} = \mathcal{L}$$

want to solve $\partial \psi(\vec{r},t) = -\underbrace{f(\vec{r}',t_r)}_{\text{source term}}$

$$G(r,r') = G(r';r) = G(r,r',t,t')$$

solve wave eq^{n} w/source

① $\mathcal{L}\, G(r,r',t,t') = -\delta(r-r')\,\delta(t-t')$

② $\psi_{soln}(\vec{r},t) = \int_{V'}\int_{t'} G(r,r',t,t')\cdot f(\vec{r},t')\cdot d\tau'\,dt'$

solve ① & plug into ②

go into phase space mode: $\psi(\vec{r},t) = \int_{-\infty}^{\infty}\psi_\omega(\vec{r},\omega)\,e^{-i\omega t}\,d\omega$

$f(\vec{r},t) = \int_{-\infty}^{\infty} f_\omega(\vec{r},\omega)\,e^{-i\omega t}\,d\omega$

if we wanna go back to regular mode: $\psi_\omega(\vec{r},\omega) = \int_{-\infty}^{\infty}\psi(\vec{r},t)\,e^{i\omega t}\,dt$

$f_\omega(\vec{r},\omega) = \int_{-\infty}^{\infty} f(\vec{r},t)\,e^{i\omega t}\,dt$

plug into $\mathcal{L}\psi = f$ wave eq^{n}

$$\left(\nabla^2 - \frac{1}{c^2}\partial_t^{\,2}\right)\int_{-\infty}^{\infty}\psi_\omega(\vec{r},\omega)\,e^{-i\omega t}\,d\omega = -\int_{-\infty}^{\infty} f_\omega(\vec{r},\omega)\,e^{-i\omega t}\,d\omega$$

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\int_{-\infty}^{\infty}\psi_\omega(\vec{r},\omega)\,e^{-i\omega t}\,d\omega = -\int_{-\infty}^{\infty} f_\omega(\vec{r},\omega)\,e^{-i\omega t}\,d\omega$$

$$\cancel{\int_{-\infty}^{\infty} e^{-i\omega t}\,d\omega}\cdot(\nabla^2 + k^2)\,\psi_\omega(\vec{r},\omega) = \cancel{\int_{-\infty}^{\infty} e^{-i\omega t}\,d\omega}\cdot -f_\omega(\vec{r},\omega)$$

Helmholtz Eq^{n}: $(\nabla^2 + k^2)\,\psi_\omega(\vec{r},\omega) = -f_\omega(\vec{r},\omega)$

→ in frequency domain

took away time so all we care about now is: $(\nabla^2 + k^2)\,G_k(\vec{r},\vec{r}',t,t') = -\delta(\vec{r}-\vec{r}')\,\delta(t-t')$

can note that this eq^{n} must be spherically symmetric → satisfies spherical wave eq^{n}

$$\frac{1}{R}\cdot\partial_R^{\,2}(R\cdot G_k) + k^2\cdot G_k = -\delta(R)$$

$$R\cdot G_k = \frac{1}{4\pi}\left(\underset{\text{outgoing}}{A e^{ikR}} + \underset{\text{ingoing}}{B e^{-ikR}}\right)$$

$$G_k(\vec{r},\vec{r}') = \underbrace{\frac{A e^{ikR}}{4\pi R}}_{\equiv\, G_k^{+}(\vec{r},\vec{r}')} + \underbrace{\frac{B e^{-ikR}}{4\pi R}}_{\equiv\, G_k^{-}(\vec{r},\vec{r}')}$$

time to go back to regular mode for time

$$G^{\pm}(r,\vec{r}';t,t') = \int_{-\infty}^{\infty} \frac{A_i e^{\pm ikR}}{4\pi R} \cdot e^{-i\omega(t-t')} \, d\omega' \quad \text{from } \delta(t-t')$$

$$\exp[\pm ikR]\exp[-i\omega(t-t')] = \exp[\pm ikR - i\omega(t-t')]$$

$$= \exp[-i\omega(t-t' \mp R/c)]$$

$$G(\vec{r},\vec{r}';t,t') = \int_{-\infty}^{\infty} d\omega \left(\frac{A}{4\pi R} e^{-i\omega(t-t'-R/c)} + \frac{B}{4\pi R} e^{-i\omega(t-t'+R/c)} \right)$$

$$= \frac{A}{4\pi R} \int_{-\infty}^{\infty} d\omega \, e^{-i\omega(t-t'-R/c)} + \frac{B}{4\pi R} \int_{-\infty}^{\infty} d\omega \, e^{-i\omega(t-t'+R/c)}$$

$$G(\vec{r},\vec{r}';t,t') = A/4\pi R \cdot \delta(t'-t_r) + \frac{B}{4\pi R} \cdot \delta(t'-t_a)$$

$t_r \equiv t - R/c$ retarded time

$t_a \equiv t + R/c$ advanced time

now plug G into ②

$$\psi_{soln}(\vec{r},t) = \int G(\vec{r},\vec{r}';t,t') \cdot f(\vec{r}',t) \cdot d\tau' \, dt'$$

$$\rightarrow \varphi(\vec{r},t) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\vec{r}',t_r)}{R} \, d\tau'$$

$$\rightarrow \vec{A}(\vec{r},t) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\vec{J}(\vec{r}',t_r)}{R} \, d\tau'$$

$$\vec{E} = -\nabla\varphi - \partial_t \vec{A}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$